

Interpolation of Morrey-Campanato and Related
Smoothness SpacesYUAN Wen¹, SICKEL Winfried² & YANG Dachun^{1,*}¹*School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems,
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Received March 10, 2015; accepted June 5, 2015; published online June ??, 2015

Abstract In this article, the authors study the interpolation of Morrey-Campanato spaces and some smoothness spaces based on Morrey spaces, e. g., Besov-type and Triebel-Lizorkin-type spaces. Various interpolation methods, including the complex method, the \pm -method and the Peetre-Gagliardo method, are studied in such a framework. Special emphasize is given to the quasi-Banach case and to the interpolation property.

Keywords Morrey space, Campanato space, Besov-type space, Triebel-Lizorkin-type space, Besov-Morrey space, Triebel-Lizorkin-Morrey space, local Morrey-type space, real and complex interpolation, \pm -method of interpolation, Peetre-Gagliardo interpolation, Calderón product, quasi-Banach lattice

MSC(2010) 46B70, 46E35

Citation: YUAN W, SICKEL W, YANG D. Interpolation of Morrey-Campanato and related smoothness spaces. *Sci China Math*, 2016, 59, doi: 10.1007/s11425-015-5047-8

1 Introduction

In this article, we try to give an overview on the interpolation of Morrey-Campanato spaces as well as the interpolation of smoothness spaces built on Morrey spaces (such as Besov-Morrey spaces, Triebel-Lizorkin-Morrey spaces, Besov-type spaces and Triebel-Lizorkin-type spaces). Special attention is paid to the quasi-Banach case and to the interpolation property.

Morrey spaces can be understood as a replacement (or a generalization) of the Lebesgue spaces $L_p(\mathbb{R}^n)$. This is immediate in view of their definitions. Let $0 < p \leq u \leq \infty$. Then the *Morrey space* $\mathcal{M}_p^u(\mathbb{R}^n)$ is defined as the collection of all p -locally Lebesgue-integrable functions f on \mathbb{R}^n such that

$$\|f\|_{\mathcal{M}_p^u(\mathbb{R}^n)} := \sup_B |B|^{1/u-1/p} \left[\int_B |f(x)|^p dx \right]^{1/p} < \infty, \quad (1.1)$$

where the supremum is taken over all balls B in \mathbb{R}^n . Obviously, $\mathcal{M}_p^p(\mathbb{R}^n) = L_p(\mathbb{R}^n)$. It is well known that the Morrey spaces have a lot of applications in partial differential equations and boundedness of operators; see, for example, Taylor [93], Kozono and Yamazaki [41], Mazzucato [59, 60] and Lemarié-Rieusset [44–46, 48]. Recently, some applications of Morrey spaces in harmonic analysis and potential

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analysis were presented in a series of papers by Adams and Xiao [1–4]. It is well known that the real-variable theory of function spaces, including Morrey spaces, and its various applications in analysis are central topics of harmonic analysis; see, for example, [27, 52, 63, 85, 86, 91, 94, 96–98, 100, 101, 106, 113–115].

The study of interpolation properties of Morrey spaces started with the articles of Stampacchia [90] in 1964 and of Campanato and Murthy [18] in 1965. They proved that, if T is a linear operator which is bounded from $L_{q_0}(\mathbb{R}^n)$ to the Morrey space $\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n)$ with operator norm M_0 and from $L_{q_1}(\mathbb{R}^n)$ to the Morrey space $\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$ with operator norm M_1 , then T is also bounded from $L_q(\mathbb{R}^n)$ to $\mathcal{M}_p^u(\mathbb{R}^n)$, where $\Theta \in (0, 1)$, $p_0, p_1, u_0, u_1, q_0, q_1 \in [1, \infty)$,

$$\frac{1}{q} := \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}, \quad \frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}, \quad \frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \quad (1.2)$$

and the operator norm satisfies

$$\|T\|_{L_q(\mathbb{R}^n) \rightarrow \mathcal{M}_p^u(\mathbb{R}^n)} := \sup_{\|f\|_{L_q(\mathbb{R}^n)} \leq 1} \|Tf\|_{\mathcal{M}_p^u(\mathbb{R}^n)} \leq M_0^{1-\Theta} M_1^\Theta.$$

Also, Spanne [89] in 1966 and Peetre [66] in 1969 gave proofs of these properties and, in addition, they discussed some generalizations via replacing the couple $(L_{q_0}(\mathbb{R}^n), L_{q_1}(\mathbb{R}^n))$ by an abstract interpolation couple (X_0, X_1) . Implicitly contained is the following assertion: Letting F be an interpolation functor of exponent Θ such that

$$F(L_{p_0}(\mathbb{R}^n), L_{p_1}(\mathbb{R}^n)) \hookrightarrow L_p(\mathbb{R}^n), \quad (1.3)$$

if T is a linear operator such that T is bounded from X_0 to $\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n)$ with norm M_0 and from X_1 to $\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$ with norm M_1 , then T is also bounded from $F(X_0, X_1)$ to $\mathcal{M}_p^u(\mathbb{R}^n)$, where u and p are defined as in (1.2) and

$$\|T\|_{F(X_0, X_1) \rightarrow \mathcal{M}_p^u(\mathbb{R}^n)} \leq c M_0^{1-\Theta} M_1^\Theta$$

with c being a non-negative constant (depending on (1.3)).

However, many questions have been left open. We mention the following:

- What about the converse of the above described property? That is, if the linear operator T is bounded from $\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n)$ into $L_{q_0}(\mathbb{R}^n)$ and from $\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$ into $L_{q_1}(\mathbb{R}^n)$, does it follow that T is also bounded from $\mathcal{M}_p^u(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$? Here $\Theta, p_0, p_1, u_0, u_1, q_0, q_1, p, q$ are as in (1.2).
- Is there any concrete interpolation method (having the interpolation property) such that the application of this method to the couple $(\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n))$ yields a Morrey space?

In 1995, Ruiz and Vega [72] gave a partial negative answer to both questions. They showed that, when $n \in \mathbb{N} \setminus \{1\}$, $\Theta \in (0, 1)$, $u \in (0, n)$, and

$$1 \leq p_1 < p_2 < \frac{n-1}{u} < p_0 < \infty,$$

then, for any given positive number C , there exists a positive continuous linear operator $T: \mathcal{M}_{p_i}^u(\mathbb{R}^n) \rightarrow L_1(\mathbb{R}^n)$, $i \in \{0, 1, 2\}$, with operator norm satisfying that $\|T\|_{\mathcal{M}_{p_i}^u(\mathbb{R}^n) \rightarrow L_1(\mathbb{R}^n)} \leq K_i$, $i \in \{0, 1\}$, but

$$\|T\|_{\mathcal{M}_{p_2}^u(\mathbb{R}^n) \rightarrow L_1(\mathbb{R}^n)} > CK_0^{1-\Theta} K_1^\Theta \quad \text{with} \quad \frac{1}{p_2} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}.$$

This explicit construction requires dimension $n > 1$. In 1999, Blasco, Ruiz and Vega [8] considered also the case $n = 1$. For a particular u , satisfying $1 < p_0 < p_1 < u < \infty$, they proved that there exist $q_0, q_1 \in (1, \infty)$ and a positive linear operator T , which is bounded from $\mathcal{M}_{p_i}^u(\mathbb{R})$ to $L_{q_i}(\mathbb{R})$, $i \in \{0, 1\}$, but not bounded from $\mathcal{M}_p^u(\mathbb{R})$ to $L_q(\mathbb{R})$, where $\Theta, p_0, p_1, q_0, q_1, p, q$ are as in (1.2). These counterexamples are making clear that, in general, the answer to the above two questions must be negative.

After the articles [72] and [8] had appeared, the believe in positive results in this area was not very big. However, the recent articles by Lemarié-Rieusset [46, 47], Yang et al. [111] and Lu et al. [56] indicated some essential progress. Lemarié-Rieusset [46] proved that, if

$$1 < p_i \leq u_i < \infty, \quad i \in \{0, 1\}, \quad \frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}, \quad \frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}, \quad (1.4)$$

then

$$[\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)]_{\Theta} \neq \mathcal{M}_p^u(\mathbb{R}^n) \quad (1.5)$$

whenever

$$p_0/u_0 \neq p_1/u_1, \quad (1.6)$$

giving a much better understanding of the negative results in this way. Furthermore, Lemarié-Rieusset [47] proved that, if (1.6) does not holds true, namely, if

$$p_0/u_0 = p_1/u_1, \quad (1.7)$$

then

$$[\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)]_{\Theta}^{\Theta} = \mathcal{M}_p^u(\mathbb{R}^n). \quad (1.8)$$

Here $[A_0, A_1]_{\Theta}$ and $[A_0, A_1]_{\Theta}^{\Theta}$ denote two different complex methods of interpolation theory introduced by Calderón [13], respectively. The restriction (1.7) will be always present throughout this article in connection with positive results. Whenever we are able to prove an interpolation formula for Morrey spaces with different p or different u , this restriction (1.7) will be used. In this particular case, we will supplement (1.4) by showing that (1.7) is necessary. The first positive results in interpolation of Morrey spaces go back to Yang, Yuan and Zhuo [111], in which they proved that

$$\begin{aligned} [\dot{\mathcal{M}}_{p_0}^{u_0}(\mathbb{R}^n), \dot{\mathcal{M}}_{p_1}^{u_1}(\mathbb{R}^n)]_{\Theta} &= [\dot{\mathcal{M}}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)]_{\Theta} \\ &= [\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \dot{\mathcal{M}}_{p_1}^{u_1}(\mathbb{R}^n)]_{\Theta} = \dot{\mathcal{M}}_p^u(\mathbb{R}^n), \end{aligned}$$

if the restrictions in (1.4) and (1.7) are satisfied. Here $\dot{\mathcal{M}}_p^u(\mathbb{R}^n)$ denotes the *closure* of the Schwartz functions in $\mathcal{M}_p^u(\mathbb{R}^n)$. In case $p_0/u_0 \neq p_1/u_1$, but p_0, p_1, u_0, u_1, p, u as in (1.4), one knows at least the following hold true:

$$[\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)]_{\Theta} \hookrightarrow [\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n)]^{1-\Theta} [\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)]^{\Theta} \hookrightarrow \mathcal{M}_p^u(\mathbb{R}^n),$$

where $X_0^{1-\Theta} X_1^{\Theta}$ denotes the Calderón product of X_0 and X_1 (see [56, 111]). Concerning the real interpolation method with u and p as in (1.4), one knows that

$$(\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n))_{\Theta, p} \hookrightarrow \mathcal{M}_p^u(\mathbb{R}^n)$$

(see [46], [60], [86]), and

$$\mathcal{M}_p^u(\mathbb{R}^n) \hookrightarrow (\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n))_{\Theta, \infty} \iff p_0/u_0 = p_1/u_1$$

(see [46]). It will be our aim to supplement these assertions. Going back to the two questions asked above, we see that they are partially answered only. We will return to these problems in Subsection 2.7.

Now we turn to Campanato spaces, which are some generalizations of Morrey spaces. We need a few notation. Let $B(x, r)$ denote the ball in \mathbb{R}^n with center in x and radius $r \in (0, \infty)$. By \mathcal{P}_k , we denote the *class of polynomials in \mathbb{R}^n of order at most k* . In addition, we put $\mathcal{P}_{-1} := \{0\}$. Let $p \in (0, \infty)$, $k \in \{-1, 0\} \cup \mathbb{N}$, $\lambda \in [0, \infty)$ and Ω be a bounded open subset of \mathbb{R}^n . Then the *Campanato space* $\mathcal{L}_k^{p, \lambda}(\Omega)$ is defined as the set of all $f \in L_p^{\text{loc}}(\Omega)$ such that

$$\|f\|_{\mathcal{L}_k^{p, \lambda}(\Omega)} := \sup_{x \in \Omega} \sup_{r > 0} \left[\frac{1}{|B(x, r) \cap \Omega|^{\lambda/n}} \inf_{P \in \mathcal{P}_k} \int_{B(x, r) \cap \Omega} |f(y) - P(y)|^p dy \right]^{1/p} < \infty;$$

see Campanato [14–17] or the more recent survey by Rafeiro et al. [70]. Here and hereafter, $L_p^{\text{loc}}(\Omega)$ denotes the *set of all locally integrable functions on Ω* . There exist various possibilities to extend this definition to \mathbb{R}^n . We decide to use the following “locally uniform” point of view (i. e., we consider a space defined with respect to balls of volume ≤ 1), picked up from Triebel [100, 3.1.1] (but we do not follow his notation).

Let $p \in (0, \infty)$, $\tau \in [0, \infty)$ and suppose, for the integer $k \geq -1$, that $k+1 > n(\tau - 1/p)$. Then $\mathcal{L}_p^\tau(\mathbb{R}^n)$ is defined as the collection of all $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ such that

$$\begin{aligned} \|f\|_{\mathcal{L}_p^\tau(\mathbb{R}^n)}^k &:= \sup_{x \in \mathbb{R}^n} \sup_{0 < r \leq 1} \left[\int_{B(x,r)} |f(y)|^p dy \right]^{1/p} \\ &+ \sup_{x \in \mathbb{R}^n} \sup_{0 < r \leq 1} |B(x,r)|^{-\tau} \left[\inf_{P \in \mathcal{P}_k} \int_{B(x,r)} |f(y) - P(y)|^p dy \right]^{1/p} < \infty. \end{aligned}$$

The space $\mathcal{L}_p^\tau(\mathbb{R}^n)$ is quasi-Banach and independent of the chosen admissible k ; see [100, 3.1.2/Theorem 3.4] (which itself is based on [97, 5.3.3] and [11, 2.1]). The following assertions are part of the classical theory of Campanato spaces:

- (a) Let $p \in (0, \infty)$ and $\tau \in [0, 1/p)$. Then, with $1/u := 1/p - \tau$, we have $\mathcal{L}_p^\tau(\mathbb{R}^n) = M_{p,\text{unif}}^u(\mathbb{R}^n)$ in the sense of equivalent quasi-norms, i. e., for all $k \geq -1$, there exist positive constants A and B such that

$$A \|f\|_{M_{p,\text{unif}}^u(\mathbb{R}^n)} \leq \|f\|_{\mathcal{L}_p^\tau(\mathbb{R}^n)}^k \leq B \|f\|_{M_{p,\text{unif}}^u(\mathbb{R}^n)}$$

for all $f \in \mathcal{L}_p^\tau(\mathbb{R}^n)$. Here the definition of $M_{p,\text{unif}}^u(\mathbb{R}^n)$ is obtained from (1.1) by restricting the supremum to balls with volume ≤ 1 .

- (b) Let $p \in (0, \infty)$ and $\tau \in (1/p, \infty)$. Then $\mathcal{L}_p^\tau(\mathbb{R}^n) = B_{\infty,\infty}^{n(\tau-1/p)}(\mathbb{R}^n)$ in the sense of equivalent quasi-norms (for all admissible k).
- (c) Let $p \in (0, \infty)$ and $\tau = 1/p$. Then, with $k = 0$, $\mathcal{L}_p^\tau(\mathbb{R}^n) = \text{bmo}(\mathbb{R}^n)$ in the sense of equivalent quasi-norms. If $p \in [1, \infty)$, then this result is true for all $k \geq 0$.

For the first two items, we refer the reader to Campanato [15], Kufner et al. [43, 4.3], Pick et al. [69, 5.3, 5.7], Brudnij [11, 2.1], and Triebel [97, 5.3.3], [100, 3.1.2/Theorem 3.4]. Concerning the last item, we refer the reader to John and Nirenberg [35] ($p \in [1, \infty)$), Long and Yang [53] ($p \in (0, 1)$) and Triebel [97, 5.3.3], [100, 3.1.2/Theorem 3.4]; see also Bourdaud [9].

The above quoted articles of Stampacchia [90], Campanato and Murthy [18], Spanne [89] and Peetre [66] have already dealt with Campanato spaces. The assertion described above with Morrey spaces remains true for the more general case of Campanato spaces (always with $p \in [1, \infty)$). We did not find more recent references for the interpolation of Campanato spaces.

Based on the recent progress in the understanding of the interpolation of Morrey spaces, we found it desirable to summarize what is known today about the interpolation of Morrey-Campanato spaces and smoothness spaces built on Morrey spaces. In almost all applications of interpolation theory, the associated boundedness problem for pairs of linear operators plays a role. For this reason, we will take care also of this circle of problems.

In the last two decades, partly due to the study of Navier-Stokes equations, there is an increasing interest in the construction of smoothness spaces based on Morrey spaces (in what follows called Morrey-type spaces); see, for example, [31, 32, 41, 49, 60, 92, 100, 101, 107, 108, 115]. For us of certain interest are the following:

- Besov-Morrey spaces $\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)$ (introduced and studied by Kozono, Yamazaki [41] and later by Mazzucato [60]);
- Triebel-Lizorkin-Morrey spaces $\mathcal{E}_{u,p,q}^s(\mathbb{R}^n)$ (introduced by Tang and Xu [92]);

- Besov-type spaces $B_{p,q}^{s,\tau}(\mathbb{R}^n)$ (introduced by El Baraka [22–24]; see also [107, 108]);
- Triebel-Lizorkin-type spaces $F_{p,q}^{s,\tau}(\mathbb{R}^n)$ (introduced in [107, 108]).
- Local function spaces $\mathcal{L}^r A_{p,q}^s(\mathbb{R}^n)$ (see the recent monographs of Triebel [100, 101]).

First attempts to a systematic investigation of all these scales are made in [115] and [85, 86]; see also [50, 110].

Plan of the article

In this article, we shall consider the interpolation properties of all of these spaces. It turns out that the \pm -method of Gustavsson and Peetre represents the most helpful tool in the interpolation of Morrey-Campanato and Morrey-type spaces. The main results with respect to this method are contained in Theorems 2.12 and Corollaries 2.14, 2.15 (see Subsection 2.2). In addition, we shall study the complex method of interpolation theory (see Subsection 2.4) and a further interpolation method, due to Gagliardo and Peetre (see Subsection 2.3). All three methods are closely related. However, it is of certain interest that, for the spaces under consideration in almost all cases, the \pm -method leads to a different result than the other two methods. In Subsection 2.3, we shall show that, in the most interesting cases, namely, the interpolation of Morrey spaces, one has to introduce new spaces; see Theorem 2.40. The determination of the interpolation spaces will be always connected with certain density questions, which will be studied in great detail in this subsection. In Subsection 2.4, we consider the complex method as well as the inner complex method. We derive the results for the (inner) complex method by tracing it back to corresponding statements for the Peetre-Gagliardo method; see Corollary 2.65. In that way, we also obtain the most complete collection of results concerning the (inner) complex interpolation of Besov and Triebel-Lizorkin spaces; see Subsection 2.4.3, in particular Theorem 2.66. Later we shall summarize the known results concerning the real method (see Subsection 2.6). In Subsection 2.7, we discuss consequences for the interpolation property. This means, our main results are described in Section 2. Interpolation of the spaces $\mathcal{L}^r A_{p,q}^s(\mathbb{R}^n)$ is discussed shortly in Section 3. Proofs are concentrated in Section 4 and given in the lexicographic order. Definitions and some basic properties of all these scales of function spaces will be given in Appendix at the end of this article.

Notation

As usual, \mathbb{N} denotes the natural numbers $\{1, 2, \dots\}$ and $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. \mathbb{Z} denotes the integers and \mathbb{R} the real numbers. The letter n is always reserved for the dimension in \mathbb{Z}^n and \mathbb{R}^n . Let $\mathcal{S}(\mathbb{R}^n)$ be the *space of all Schwartz functions* on \mathbb{R}^n endowed with the classical topology and $\mathcal{S}'(\mathbb{R}^n)$ its *topological dual space*, namely, the set of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$ endowed with the weak-* topology. We also need the *Fourier transform* \widehat{f} , which is defined on the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions. We denote, by $C_c^\infty(\mathbb{R}^n)$, the collection of all complex-valued infinitely differentiable functions with compact support.

If X and Y are two quasi-Banach spaces, then the symbol $X \hookrightarrow Y$ indicates that the embedding of X into Y is continuous. By $\mathcal{L}(X, Y)$, we denote the collection of all linear and bounded operators $T : X \rightarrow Y$, equipped with the quasi-norm

$$\|T\|_{\mathcal{L}(X,Y)} := \|T\|_{X \rightarrow Y} := \sup_{\|x\|_X \leq 1} \|Tx\|_Y.$$

The *symbol* C denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line, and the *symbol* $C_{(\alpha, \dots)}$ denotes a positive constant depending on the parameters α, \dots . The *symbol* $A \lesssim B$ means that: there exists a positive constant C such that $A \leq CB$. Similarly \gtrsim is defined. The symbol $A \asymp B$ will be used as an abbreviation of $A \lesssim B \lesssim A$.

Let

$$\mathcal{Q} := \{Q_{j,k} := 2^{-j}([0, 1)^n + k) : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$$

be the collection of all *dyadic cubes* in \mathbb{R}^n . We also need to consider the subset

$$\mathcal{Q}^* := \{Q_{j,k} : j \in \mathbb{Z}_+, k \in \mathbb{Z}^n\}.$$

For all $Q \in \mathcal{Q}$, let $j_Q := -\log_2 \ell(Q)$, where $\ell(Q)$ denotes the *side-length* of the cube.

Convention. If there is no reason to distinguish between $B_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $F_{p,q}^{s,\tau}(\mathbb{R}^n)$ (resp. between the corresponding sequence spaces $b_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $f_{p,q}^{s,\tau}(\mathbb{R}^n)$), we simply write $A_{p,q}^{s,\tau}(\mathbb{R}^n)$ (resp. $a_{p,q}^{s,\tau}(\mathbb{R}^n)$). Here we always assume $p < \infty$ in case $A_{p,q}^{s,\tau}(\mathbb{R}^n) = F_{p,q}^{s,\tau}(\mathbb{R}^n)$ (resp. in case $a_{p,q}^{s,\tau}(\mathbb{R}^n) = f_{p,q}^{s,\tau}(\mathbb{R}^n)$). The same convention applies with respect to the spaces $B_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$ and $F_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$ (resp. $A_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$) as well as to $b_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$ and $f_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$ (resp. $a_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$).

2 Various interpolation methods

Nowadays interpolation theory represents an important tool in various branches of mathematics. Consulting the most quoted monographs on interpolation theory (see [5, 7, 42, 57, 68, 94]), one obtains the impression that the real and the complex methods are most important. In the context of Morrey spaces $\mathcal{M}_p^u(\mathbb{R}^n)$, the situation turns out to be different. The most useful interpolation method, in case of different p (and/or different u), turns out to be the \pm -method of Gustavsson and Peetre [30]; see also Ovchinnikov [65]. The real method is the most helpful in those situations where we fix p and u (or p and τ). Also the complex method as well as a further method, introduced by Peetre [67], but based on some earlier work of Gagliardo [28] and, in what follows, called Peetre-Gagliardo interpolation method, will be studied.

The main tool in this article will be the Calderón product. Our method heavily depends on the articles by Nilsson [64] (relations between the Calderón product and other interpolation methods in the abstract setting of quasi-Banach spaces) and by Yang et al. [111] (concrete Calderón products). In all cases, special emphasize is given to the interpolation property.

2.1 The Calderón product

Let $(\mathfrak{X}, \mathcal{S}, \mu)$ be a σ -finite measure space and let \mathfrak{M} be the class of all complex-valued, μ -measurable functions on \mathfrak{X} . Then a quasi-Banach space $X \subset \mathfrak{M}$ is called a *quasi-Banach lattice of functions* if, for every $f \in X$ and $g \in \mathfrak{M}$ with $|g(x)| \leq |f(x)|$ for μ -a.e. $x \in \mathfrak{X}$, one has $g \in X$ and $\|g\|_X \leq \|f\|_X$.

Definition 2.1. Let $X_j \subset \mathfrak{M}$, $j \in \{0, 1\}$, be quasi-Banach lattices of functions, and $\Theta \in (0, 1)$. Then the *Calderón product* $X_0^{1-\Theta} X_1^\Theta$ of X_0 and X_1 is defined as the collection of all functions $f \in \mathfrak{M}$ such that the *quasi-norm*

$$\|f\|_{X_0^{1-\Theta} X_1^\Theta} := \inf \left\{ \|f_0\|_{X_0}^{1-\Theta} \|f_1\|_{X_1}^\Theta : |f| \leq |f_0|^{1-\Theta} |f_1|^\Theta \quad \mu\text{-a.e.}, f_j \in X_j, j \in \{0, 1\} \right\}$$

is finite.

Remark 2.2. Calderón products were introduced by Calderón [13, 13.5]. The usefulness of this method and its limitations have been perfectly described by Frazier and Jawerth [26] which we quote now: *Although restricted to the case of a lattice, the Calderón product has the advantage of being defined in the quasi-Banach case, and, frequently, of being easy to compute. It has the disadvantage that the interpolation property (i. e., the property that a linear transformation T bounded on X_0 and X_1 should be bounded on the spaces in between) is not clear in general.*

Calderón products have proved to be a very useful tool for the study of various interpolation methods; see, for example, [13, 40, 55, 64]. We collect a few useful properties of Calderón products for the later use; see, for example, [111].

Lemma 2.3. Let $X_j \subset \mathfrak{M}$, $j \in \{0, 1\}$, be quasi-Banach lattices of functions and let $\Theta \in (0, 1)$.

(i) Then the Calderón product $X_0^{1-\Theta} X_1^\Theta$ is a quasi-Banach space.

(ii) Define $\widetilde{X_0^{1-\Theta} X_1^\Theta}$ as the collection of all f such that there exist a positive real number λ and elements $g \in X_0$ and $h \in X_1$ satisfying

$$|f| \leq \lambda |g|^{1-\Theta} |h|^\Theta \text{ a.e., } \|g\|_{X_0} \leq 1 \text{ and } \|h\|_{X_1} \leq 1.$$

Let

$$\|f\|_{\widetilde{X_0^{1-\Theta} X_1^\Theta}} := \inf \left\{ \lambda > 0 : |f| \leq \lambda |g|^{1-\Theta} |h|^\Theta \text{ a.e., } \|g\|_{X_0} \leq 1 \text{ and } \|h\|_{X_1} \leq 1 \right\}.$$

Then $\widetilde{X_0^{1-\Theta} X_1^\Theta} = X_0^{1-\Theta} X_1^\Theta$ in the sense of equivalent quasi-norms.

Now we turn to the investigation of linear operators and Calderón products. An operator T on a quasi-Banach lattice X is said to be *positive* if $Tf \geq 0$ whenever $f \geq 0$ is in its domain. In 1990, Frazier and Jawerth [26, Proposition 8.1] obtained the following result; see also Shestakov [84, Theorem 3.1] for the Banach space case.

Proposition 2.4. Let $\Theta \in (0, 1)$. Let X_i and Y_i be quasi-Banach lattices and let T be a positive linear operator bounded from X_i to Y_i , $i \in \{0, 1\}$. Then T is bounded considered as a mapping from the Calderón product $X_0^{1-\Theta} X_1^\Theta$ to the Calderón product $Y_0^{1-\Theta} Y_1^\Theta$ and

$$\|T\|_{X_0^{1-\Theta} X_1^\Theta \rightarrow Y_0^{1-\Theta} Y_1^\Theta} \leq \|T\|_{X_0 \rightarrow Y_0}^{1-\Theta} \|T\|_{X_1 \rightarrow Y_1}^\Theta.$$

Notice that, in Proposition 2.4, we need the restriction that the operator T is positive. The Calderón product is not an interpolation construction in the class of Banach function lattices. Indeed, an example to show this was given by Lozanovskii [54].

We are interested in concrete realizations. It is easy to see that Morrey spaces are quasi-Banach lattices, but the smoothness spaces $B_{p,q}^{s,\tau}(\mathbb{R}^n)$, $F_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)$ might not be (at least in general).

Theorem 2.5. Let $\Theta \in (0, 1)$, $0 < p_0 \leq u_0 < \infty$ and $0 < p_1 \leq u_1 < \infty$ such that

$$\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \quad \text{and} \quad \frac{1}{u} = \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}.$$

(i) It holds true that

$$[\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n)]^{1-\Theta} [\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)]^\Theta \hookrightarrow \mathcal{M}_p^u(\mathbb{R}^n).$$

(ii) If $u_0 p_1 = u_1 p_0$, then

$$[\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n)]^{1-\Theta} [\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)]^\Theta = \mathcal{M}_p^u(\mathbb{R}^n).$$

(iii) If $u_0 p_1 \neq u_1 p_0$, then

$$[\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n)]^{1-\Theta} [\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)]^\Theta \subsetneq \mathcal{M}_p^u(\mathbb{R}^n).$$

For proofs of (i) and (ii) of Theorem 2.5, we refer the reader to Lu et al. [56]. Part (ii) of Theorem 2.5 is explicitly stated therein, and part (i) of Theorem 2.5 can be found in [56, Formula (2.3)]. Part (iii) of Theorem 2.5 follows from [46] (see Subsection 4.1 for more details).

The identity in (ii) of Theorem 2.5 in case $u_i = p_i$, $i \in \{0, 1\}$, given by

$$[L_{p_0}(\mathbb{R}^n)]^{1-\Theta} [L_{p_1}(\mathbb{R}^n)]^\Theta = L_p(\mathbb{R}^n),$$

can be found in several places, we refer the reader to [12, Exercise 4.3.8], [42, Formula 1.6.1], [58, P. 179, Exercise 3] (Banach case) and [87] (general situation). For later use, we formulate one more elementary example as follows. Let $s \in \mathbb{R}$ and $q \in (0, \infty]$. Define $\ell_q^s(\mathbb{Z}_+)$ as the collection of all sequences $\{a_j\}_{j \in \mathbb{Z}_+} \subset \mathbb{C}$ such that

$$\|\{a_j\}_{j \in \mathbb{Z}_+}\|_{\ell_q^s(\mathbb{Z}_+)} := \left\{ \sum_{j \in \mathbb{Z}_+} 2^{jsq} |a_j|^q \right\}^{\frac{1}{q}} < \infty.$$

Lemma 2.6. Let $s_0, s_1 \in \mathbb{R}$, $p_0, p_1 \in (0, \infty]$ and $\Theta \in (0, 1)$ such that $s = (1-\Theta)s_0 + \Theta s_1$ and $\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$. Then

$$[\ell_{p_0}^{s_1}(\mathbb{Z}_+)]^{1-\Theta} [\ell_{p_1}^{s_0}(\mathbb{Z}_+)]^\Theta = \ell_p^s(\mathbb{Z}_+).$$

The proof of $[L_{p_0}(\mathbb{R}^n)]^{1-\Theta} [L_{p_1}(\mathbb{R}^n)]^\Theta = L_p(\mathbb{R}^n)$, given in [87], carries over to the discrete situation. Even more interesting are the more complicated sequence spaces $b_{p,q}^{s,\tau}(\mathbb{R}^n)$, $f_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $n_{u,p,q}^s(\mathbb{R}^n)$, associated to the scales $B_{p,q}^{s,\tau}(\mathbb{R}^n)$, $F_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)$; see Appendix at the end of this article for their definitions and properties. All these sequence spaces are quasi-Banach lattices. The following results were proved in [111, Propositions 2.6, 2.7 and 2.8] and are of basic importance for the remainder of this article.

Proposition 2.7. Let $\Theta \in (0, 1)$, $s, s_0, s_1 \in \mathbb{R}$, $\tau, \tau_0, \tau_1 \in [0, \infty)$, $p, p_0, p_1 \in (0, \infty]$ and $q, q_0, q_1 \in (0, \infty]$ such that $s = s_0(1-\Theta) + s_1\Theta$, $\tau = \tau_0(1-\Theta) + \tau_1\Theta$, $\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$ and $\frac{1}{q} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}$. Then it holds true that

$$[a_{p_0,q_0}^{s_0,\tau_0}(\mathbb{R}^n)]^{1-\Theta} [a_{p_1,q_1}^{s_1,\tau_1}(\mathbb{R}^n)]^\Theta \hookrightarrow a_{p,q}^{s,\tau}(\mathbb{R}^n), \quad a \in \{f, b\}.$$

If, in addition, $\tau_0 p_0 = \tau_1 p_1$, then

$$[a_{p_0,q_0}^{s_0,\tau_0}(\mathbb{R}^n)]^{1-\Theta} [a_{p_1,q_1}^{s_1,\tau_1}(\mathbb{R}^n)]^\Theta = a_{p,q}^{s,\tau}(\mathbb{R}^n), \quad a \in \{f, b\}.$$

Proposition 2.8. Let $\Theta \in (0, 1)$, $s, s_0, s_1 \in \mathbb{R}$, $q, q_0, q_1 \in (0, \infty]$, $0 < p \leq u \leq \infty$, $0 < p_0 \leq u_0 \leq \infty$ and $0 < p_1 \leq u_1 \leq \infty$ such that $\frac{1}{q} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}$, $\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$, $s = s_0(1-\Theta) + s_1\Theta$ and $\frac{1}{u} = \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}$. Then it holds true that

$$[n_{u_0,p_0,q_0}^{s_0}(\mathbb{R}^n)]^{1-\Theta} [n_{u_1,p_1,q_1}^{s_1}(\mathbb{R}^n)]^\Theta \hookrightarrow n_{u,p,q}^s(\mathbb{R}^n).$$

If, in addition, $p_0 u_1 = p_1 u_0$, then

$$[n_{u_0,p_0,q_0}^{s_0}(\mathbb{R}^n)]^{1-\Theta} [n_{u_1,p_1,q_1}^{s_1}(\mathbb{R}^n)]^\Theta = n_{u,p,q}^s(\mathbb{R}^n).$$

The assertions stated in Propositions 2.7 and 2.8 are far away from being trivial. The prototype is the ingenious proof of the formula

$$[f_{p_0,q_0}^{s_0,0}(\mathbb{R}^n)]^{1-\Theta} [f_{p_1,q_1}^{s_1,0}(\mathbb{R}^n)]^\Theta = f_{p,q}^{s,0}(\mathbb{R}^n),$$

due to Frazier and Jawerth [26]; see also Bownik [10]. Various different proofs of

$$[b_{p_0,q_0}^{s_0,0}(\mathbb{R}^n)]^{1-\Theta} [b_{p_1,q_1}^{s_1,0}(\mathbb{R}^n)]^\Theta = b_{p,q}^{s,0}(\mathbb{R}^n)$$

can be found in the literatures, we refer the reader to Mendes and Mitrea [61], Kalton et al. [39] and Sickel et al. [87].

2.2 The \pm -method of Gustavsson and Peetre

The next interpolation method, called the \pm -method, was originally introduced by Gustavsson and Peetre [29,30]. Later it has been considered also by Bereznoi [6], Gustavsson [29], Nilsson [64], Ovchinnikov [65] and Shestakov [84].

Consider a couple of quasi-Banach spaces (for short, a quasi-Banach couple), X_0 and X_1 , which are continuously embedded into a larger Hausdorff topological vector space Y . The space $X_0 + X_1$ is given by

$$X_0 + X_1 := \{h \in Y : \exists h_i \in X_i, i \in \{0, 1\}, \text{ such that } h = h_0 + h_1\},$$

equipped with the quasi-norm

$$\|h\|_{X_0+X_1} := \inf \left\{ \|h_0\|_{X_0} + \|h_1\|_{X_1} : h = h_0 + h_1, h_0 \in X_0 \text{ and } h_1 \in X_1 \right\}.$$

Definition 2.9. Let (X_0, X_1) be a quasi-Banach couple and $\Theta \in (0, 1)$. An $a \in X_0 + X_1$ is said to belong to $\langle X_0, X_1, \Theta \rangle$ if there exists a sequence $\{a_i\}_{i \in \mathbb{Z}} \subset X_0 \cap X_1$ such that $a = \sum_{i \in \mathbb{Z}} a_i$ with convergence in $X_0 + X_1$ and, for any finite subset $F \subset \mathbb{Z}$ and any bounded sequence $\{\varepsilon_i\}_{i \in \mathbb{Z}} \subset \mathbb{C}$,

$$\left\| \sum_{i \in F} \varepsilon_i 2^{i(j-\Theta)} a_i \right\|_{X_j} \leq C \sup_{i \in \mathbb{Z}} |\varepsilon_i| \quad (2.1)$$

for some non-negative constant C independent of F and $j \in \{0, 1\}$. The *quasi-norm* of $a \in \langle X_0, X_1, \Theta \rangle$ is defined as

$$\|a\|_{\langle X_0, X_1, \Theta \rangle} := \inf \{C : C \text{ satisfies (2.1)}\}.$$

We recall a few results from [30, Proposition 6.1].

Proposition 2.10. Let (A_0, A_1) and (B_0, B_1) be any two quasi-Banach couples and $\Theta \in (0, 1)$.

- (i) It holds true that $\langle A_0, A_1, \Theta \rangle$ is a quasi-Banach space.
- (ii) If $T \in \mathcal{L}(A_0, B_0) \cap \mathcal{L}(A_1, B_1)$, then T maps $\langle A_0, A_1, \Theta \rangle$ continuously into $\langle B_0, B_1, \Theta \rangle$. Furthermore,

$$\|T\|_{\langle A_0, A_1, \Theta \rangle \rightarrow \langle B_0, B_1, \Theta \rangle} \leq \max \{\|T\|_{A_0 \rightarrow B_0}, \|T\|_{A_1 \rightarrow B_1}\}.$$

Next we recall a standard method in the interpolation theory, namely, the method of retraction. Let X and Y be two quasi-Banach spaces. Then Y is called a *retract* of X if there exist two bounded linear operators $E : Y \rightarrow X$ and $R : X \rightarrow Y$ such that $R \circ E = I$, the identity map on Y . Proposition 2.10 allows to apply standard arguments to establish the following property (we refer the reader to [94, Theorem 1.2.4] for those arguments and, in addition, one should notice that the closed graph theorem remains true in the context of quasi-Banach spaces).

Proposition 2.11. Let (X_0, X_1) and (Y_0, Y_1) be two interpolation couples of quasi-Banach spaces such that Y_j is a retract of X_j , $j \in \{0, 1\}$. Then, for each $\Theta \in (0, 1)$,

$$\langle Y_0, Y_1, \Theta \rangle = R(\langle X_0, X_1, \Theta \rangle).$$

As a consequence of Propositions 2.7 and 2.8 and a general result of Nilsson [64, Theorem 2.1] (see Proposition 4.3 below), we obtain the first main result of this article.

Theorem 2.12. Let $\Theta \in (0, 1)$, $s_i \in \mathbb{R}$, $\tau_i \in [0, \infty)$, $p_i, q_i \in (0, \infty]$ and $u_i \in [p_i, \infty]$, $i \in \{0, 1\}$, such that $s = (1 - \Theta)s_0 + \Theta s_1$, $\tau = (1 - \Theta)\tau_0 + \Theta \tau_1$,

$$\frac{1}{p} = \frac{1 - \Theta}{p_0} + \frac{\Theta}{p_1}, \quad \frac{1}{q} = \frac{1 - \Theta}{q_0} + \frac{\Theta}{q_1} \quad \text{and} \quad \frac{1}{u} = \frac{1 - \Theta}{u_0} + \frac{\Theta}{u_1}.$$

- (i) If $\tau_0 p_0 = \tau_1 p_1$, then

$$\langle A_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), A_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n), \Theta \rangle = A_{p, q}^{s, \tau}(\mathbb{R}^n), \quad A \in \{B, F\}.$$

- (ii) If $p_0 u_1 = p_1 u_0$, then

$$\langle \mathcal{N}_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n), \mathcal{N}_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n), \Theta \rangle = \mathcal{N}_{u, p, q}^s(\mathbb{R}^n).$$

Remark 2.13. Here are several observations on Theorem 2.12.

(i) We comment on the restriction $\tau_0 p_0 = \tau_1 p_1$. This required identity has serious consequences. If either $\tau_0 = 0$ or $\tau_1 = 0$, it immediately follows $\tau = \tau_0 = \tau_1 = 0$ and we are back in the classical situation of Besov and Triebel-Lizorkin spaces. If $\max\{p_0, p_1\} < \infty$ and either $\tau_0 = 1/p_0$ or $\tau_1 = 1/p_1$, then we obtain

$$\tau_0 p_0 = \tau_1 p_1 = \tau p = 1.$$

With $A = F$, Theorem 2.12(i) reads as

$$\langle F_{p_0, q_0}^{s_0, 1/p_0}(\mathbb{R}^n), F_{p_1, q_1}^{s_1, 1/p_1}(\mathbb{R}^n), \Theta \rangle = \langle F_{\infty, q_0}^{s_0}(\mathbb{R}^n), F_{\infty, q_1}^{s_1}(\mathbb{R}^n), \Theta \rangle = F_{\infty, q}^s(\mathbb{R}^n);$$

see Proposition 5.2(ii). This has been known before, we refer the reader to Frazier and Jawerth [26, Theorem 8.5]. The counterpart with $A = B$ seems to be a novelty. Finally, we consider the case that $\max_{i \in \{0,1\}} \{\tau_i - 1/p_i\} > 0$. Then the above restriction implies

$$\min \{\tau - 1/p, \tau_0 - 1/p_0, \tau_1 - 1/p_1\} > 0.$$

Taking into account Proposition 5.2(ii), Theorem 2.12 reduces to

$$\begin{aligned} \langle B_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), B_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n), \Theta \rangle &= \langle B_{\infty, \infty}^{s_0+n(\tau_0-1/p_0)}(\mathbb{R}^n), B_{\infty, \infty}^{s_1+n(\tau_1-1/p_1)}(\mathbb{R}^n), \Theta \rangle \\ &= B_{\infty, \infty}^{s+n(\tau-1/p)}(\mathbb{R}^n). \end{aligned}$$

(ii) We explain the difference in the restrictions in (i) and (ii) of Theorem 2.12. For simplicity, we concentrate on the F -case. Recall that, when $\tau \in [0, 1/p)$, the spaces $F_{p,q}^{s,\tau}(\mathbb{R}^n)$ coincide with the Triebel-Lizorkin-Morrey spaces $\mathcal{E}_{u,p,q}^s(\mathbb{R}^n)$, that is,

$$\mathcal{E}_{u,p,q}^s(\mathbb{R}^n) = F_{p,q}^{s, 1/p-1/u}(\mathbb{R}^n), \quad 0 < p \leq u < \infty;$$

see Proposition 5.5 in Appendix. A reformulation of Theorem 2.12(i) (with $A = F$) reads as follows:

$$\langle \mathcal{E}_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n), \mathcal{E}_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n), \Theta \rangle = \mathcal{E}_{u, p, q}^s(\mathbb{R}^n)$$

holds true under the restrictions in Theorem 2.12(ii). The condition $\tau_0 p_0 = \tau_1 p_1$ is obviously equivalent to $p_0 u_1 = p_1 u_0$.

(iii) In the case $\tau_0 = \tau_1 = 0$, the formula

$$\langle F_{p_0, q_0}^{s_0}(\mathbb{R}^n), F_{p_1, q_1}^{s_1}(\mathbb{R}^n), \Theta \rangle = \langle F_{p_0, q_0}^{s_0, 0}(\mathbb{R}^n), F_{p_1, q_1}^{s_1, 0}(\mathbb{R}^n), \Theta \rangle = F_{p, q}^{s, 0}(\mathbb{R}^n) = F_{p, q}^s(\mathbb{R}^n) \quad (2.2)$$

was proved by Frazier and Jawerth [26, Theorem 8.5]. Also the formula

$$\langle F_{p_0, q_0}^{s_0, 1/p_0}(\mathbb{R}^n), F_{p_1, q_1}^{s_1, 1/p_1}(\mathbb{R}^n), \Theta \rangle = F_{p, q}^{s, 1/p}(\mathbb{R}^n) = F_{\infty, q}^s(\mathbb{R}^n)$$

as well as

$$\langle F_{p_0, q_0}^{s_0, 1/p_0}(\mathbb{R}^n), B_{p_1, \infty}^{s_1, 1/p_1}(\mathbb{R}^n), \Theta \rangle = F_{p, q}^{s, 1/p}(\mathbb{R}^n) = F_{\infty, q}^s(\mathbb{R}^n)$$

can be found therein. Notice that the idea used in the proof for (2.2) carries over to the general case.

(iv) As we have seen in (i) of this remark, the restriction $\tau_0 p_0 = \tau_1 p_1$ splits the admissible parameters into four groups:

- (a) $\tau_0 = \tau_1 = 0$;
- (b) $0 < \tau_i < 1/p_i$, $i \in \{0, 1\}$;
- (c) $\tau_0 - 1/p_0 = \tau_1 - 1/p_1 = 0$;
- (d) $\max_{i \in \{0,1\}} \{\tau_i - 1/p_i\} > 0$.

Our methods do not apply to other situations. However, Frazier and Jawerth [26, Theorem 8.5] proved that, if $p_0, p_1 \in (0, \infty)$ (and with no further restrictions), then

$$\langle F_{p_0, q_0}^{s_0, 0}(\mathbb{R}^n), F_{p_1, q_1}^{s_1, 1/p_1}(\mathbb{R}^n), \Theta \rangle = F_{p, q}^{s, 0}(\mathbb{R}^n)$$

and, if $p_0, p_1 \in (0, \infty)$ and $\tau_1 \in (1/p_1, \infty)$, then

$$\begin{aligned} \langle F_{p_0, q_0}^{s_0, 0}(\mathbb{R}^n), F_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n), \Theta \rangle &= \langle F_{p_0, q_0}^{s_0, 0}(\mathbb{R}^n), B_{\infty, \infty}^{s_1+n(\tau_1-1/p_1)}(\mathbb{R}^n), \Theta \rangle \\ &= F_{p, q}^{s+n(\tau-1/p)+n(1-\Theta)/p_0, 0}(\mathbb{R}^n). \end{aligned}$$

The \pm -method and Morrey-Campanato spaces

Recall that $F_{p,2}^{0,1/p-1/u}(\mathbb{R}^n) = \mathcal{M}_p^u(\mathbb{R}^n)$ if $1 < p \leq u < \infty$ (see Mazzucato [59] and Sawano [76]). Then, partially as a further corollary of Theorem 2.12, we have the following conclusion on the interpolation of Morrey spaces.

Corollary 2.14. *Let $\Theta \in (0, 1)$, $0 < p_0 \leq u_0 < \infty$ and $0 < p_1 \leq u_1 < \infty$. Let $\frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}$ and $\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$.*

(i) *If $p_0 u_1 = p_1 u_0$, then*

$$\langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n), \Theta \rangle = \mathcal{M}_p^u(\mathbb{R}^n).$$

(ii) *If $\min\{p_0, p_1\} > 1$ and $p_0 u_1 \neq p_1 u_0$, then*

$$\langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n), \Theta \rangle \neq \mathcal{M}_p^u(\mathbb{R}^n).$$

Corollary 2.14(i) has been known before; see [56, Theorem 2.3]. However, our proof, given in Subsection 4.2, will differ from that one given in [56, Theorem 2.3]. Corollary 2.14 allows us now to consider also Campanato spaces.

Corollary 2.15. *Let $\Theta \in (0, 1)$, $p_0, p_1 \in (0, \infty)$ and $\tau_0, \tau_1 \in [0, \infty)$. If $\tau := (1 - \Theta)\tau_0 + \Theta\tau_1$, $\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$ and $p_0\tau_0 = p_1\tau_1$, then*

$$\langle \mathcal{L}_{p_0}^{\tau_0}(\mathbb{R}^n), \mathcal{L}_{p_1}^{\tau_1}(\mathbb{R}^n), \Theta \rangle = \mathcal{L}_p^\tau(\mathbb{R}^n)$$

(by using the convention that, in case $\tau = 1/p$, either $p \in [1, \infty)$ and $k \geq 0$ or $p \in (0, 1)$ and $k = 0$).

2.3 The Peetre-Gagliardo interpolation method

The following interpolation method $\langle \cdot, \cdot \rangle_\Theta$ was introduced by Peetre [67] (based on some earlier work of Gagliardo [28]).

Definition 2.16. Let (X_0, X_1) be a quasi-Banach couple and $\Theta \in (0, 1)$. An $a \in X_0 + X_1$ is said to belong to $\langle X_0, X_1 \rangle_\Theta$ if there exists a sequence $\{a_i\}_{i \in \mathbb{Z}} \subset X_0 \cap X_1$ such that $a = \sum_{i \in \mathbb{Z}} a_i$ with convergence in $X_0 + X_1$ and, for any bounded sequence $\{\varepsilon_i\}_{i \in \mathbb{Z}} \subset \mathbb{C}$,

$$\sum_{i \in \mathbb{Z}} \varepsilon_i 2^{i(j-\Theta)} a_i$$

converges in X_j , $j \in \{0, 1\}$, and satisfies

$$\left\| \sum_{i \in \mathbb{Z}} \varepsilon_i 2^{i(j-\Theta)} a_i \right\|_{X_j} \leq C \sup_{i \in \mathbb{Z}} |\varepsilon_i| \quad (2.3)$$

for some non-negative constant C independent of $j \in \{0, 1\}$. The *quasi-norm* of $a \in \langle X_0, X_1 \rangle_\Theta$ is defined as

$$\|a\|_{\langle X_0, X_1 \rangle_\Theta} := \inf \{C : C \text{ satisfies (2.3)}\}.$$

Proposition 2.17. *Let (A_0, A_1) and (B_0, B_1) be any two quasi-Banach couples and $\Theta \in (0, 1)$.*

(i) *It holds true that $\langle A_0, A_1 \rangle_\Theta$ is a quasi-Banach space.*

(ii) *If $T \in \mathcal{L}(A_0, B_0) \cap \mathcal{L}(A_1, B_1)$, then T maps $\langle A_0, A_1 \rangle_\Theta$ continuously into $\langle B_0, B_1 \rangle_\Theta$. Furthermore*

$$\|T\|_{\langle A_0, A_1 \rangle_\Theta \rightarrow \langle B_0, B_1 \rangle_\Theta} \leq \max \{\|T\|_{A_0 \rightarrow B_0}, \|T\|_{A_1 \rightarrow B_1}\}.$$

By Proposition 2.17, we also have a counterpart of Proposition 2.11 as follows.

Proposition 2.18. *Let (X_0, X_1) and (Y_0, Y_1) be two interpolation couples of quasi-Banach spaces such that Y_j is a retract of X_j , $j \in \{0, 1\}$. Then, for each $\Theta \in (0, 1)$,*

$$\langle Y_0, Y_1 \rangle_\Theta = R(\langle X_0, X_1 \rangle_\Theta).$$

Of course, there is only a minimal difference between $\langle \cdot, \cdot \rangle_\Theta$ and $\langle \cdot, \cdot, \Theta \rangle$. As an immediate conclusion of their definitions, we obtain $\langle X_0, X_1 \rangle_\Theta \hookrightarrow \langle X_0, X_1, \Theta \rangle$. In some case, one can say more about the relation between $\langle X_0, X_1 \rangle_\Theta$ and $\langle X_0, X_1, \Theta \rangle$. To this end, we need a few more notation.

A quasi-Banach space X is called an *intermediate space* with respect to $X_0 + X_1$ if

$$X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1.$$

Definition 2.19. Let X_0, X_1 be a couple of quasi-Banach spaces and X an intermediate space with respect to $X_0 + X_1$. Define

$$X^\# := (X_0, X_1, X, \#) := \overline{X_0 \cap X_1}^{\|\cdot\|_X},$$

i. e., $X^\#$ is the closure of $X_0 \cap X_1$ in X .

Many times the space $\overline{X_0 \cap X_1}^{\|\cdot\|_X}$ is denoted by X° ; see, e. g., [64]. However, in this article, the closure of the test functions in X , denoted by \mathring{X} , will play a certain role. By using this different notation, we hope to avoid confusion. For us of certain interest is the following relation; see Nilsson [64, (1.5)] and also Janson [34, Theorem 1.8].

Proposition 2.20. Let (X_0, X_1) be a couple of quasi-Banach spaces and $\Theta \in (0, 1)$. Then

$$\langle X_0, X_1 \rangle_\Theta = (X_0, X_1, \langle X_0, X_1, \Theta \rangle, \#), \quad \Theta \in (0, 1).$$

Combining Theorem 2.12 and its sequence space version Theorem 4.4, Proposition 2.20, and Propositions 5.8 and 5.11 in Appendix below, we immediately obtain the next interesting conclusions.

Proposition 2.21. Let all parameters be as in Theorem 2.12.

(i) If $\tau_0 p_0 = \tau_1 p_1$, then

$$\langle A_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), A_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n) \rangle_\Theta = (A_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), A_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n), A_{p, q}^{s, \tau}(\mathbb{R}^n), \#). \quad (2.4)$$

(ii) If $p_0 u_1 = p_1 u_0$, then

$$\langle \mathcal{N}_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n), \mathcal{N}_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n) \rangle_\Theta = (\mathcal{N}_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n), \mathcal{N}_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n), \mathcal{N}_{u, p, q}^s(\mathbb{R}^n), \#). \quad (2.5)$$

Remark 2.22. The homogeneous counterparts of Theorems 2.12 and 2.21 also hold true (more exactly, Theorem 2.12 and Proposition 2.21 with the inhomogeneous spaces $A_{p, q}^{s, \tau}(\mathbb{R}^n)$ and $\mathcal{N}_{u, p, q}^s(\mathbb{R}^n)$ replaced, respectively, by their homogeneous counterparts $\dot{A}_{p, q}^{s, \tau}(\mathbb{R}^n)$ and $\dot{\mathcal{N}}_{u, p, q}^s(\mathbb{R}^n)$ remain valid). However, to limit the length of this article, we omit the details and refer the reader to [111] for some results in this direction.

Hence, we are left with the problem to calculate the right-hand sides in (2.4) and (2.5). But this seems to be a difficult problem. A first hint in this direction is given by the following observation. Therefore we need new spaces.

Definition 2.23. Let X be a quasi-Banach space of distributions or functions.

(i) By \mathring{X} we denote the closure in X of the set of all infinitely differentiable functions f such that $D^\alpha f \in X$ for all $\alpha \in (\mathbb{Z}_+)^n$.

(ii) Let $C_c^\infty(\mathbb{R}^n) \hookrightarrow X$. Then we denote by \mathring{X} the closure of $C_c^\infty(\mathbb{R}^n)$ in X .

Remark 2.24. Recall that, in Section 1 of this article, we define $\mathring{\mathcal{M}}_p^u(\mathbb{R}^n)$ as the closure of the Schwartz functions in the Morrey space $\mathcal{M}_p^u(\mathbb{R}^n)$. This definition coincides with that in Definition 2.23(ii) with $X = \mathcal{M}_p^u(\mathbb{R}^n)$. Indeed, it is obvious that the closure of Schwartz functions in $\mathcal{M}_p^u(\mathbb{R}^n)$ contains the closure of $C_c^\infty(\mathbb{R}^n)$, while the inverse inclusion follows from the well-known fact that a Schwartz function can be approximated by $C_c^\infty(\mathbb{R}^n)$ functions in Schwartz norms and hence in Morrey spaces (due to the continuous embedding from the Schwartz space into Morrey spaces).

Clearly, $\mathring{A}_{p, q}^{s, \tau}(\mathbb{R}^n) \hookrightarrow \dot{A}_{p, q}^{s, \tau}(\mathbb{R}^n)$ and $\mathring{\mathcal{N}}_{u, p, q}^s(\mathbb{R}^n) \hookrightarrow \dot{\mathcal{N}}_{u, p, q}^s(\mathbb{R}^n)$. First we clarify the relation of these two scales to each other.

Lemma 2.25. *Let $A \in \{B, F\}$ and $s \in \mathbb{R}$. Then*

- (i) $\mathring{A}_{p,q}^{s,\tau}(\mathbb{R}^n) = \mathring{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ if and only if $\tau = 0$, $p \in (0, \infty)$ and $q \in (0, \infty]$.
- (ii) $\mathring{\mathcal{N}}_{u,p,q}^s(\mathbb{R}^n) = \mathring{\mathcal{N}}_{u,p,q}^s(\mathbb{R}^n)$ if and only if $u = p \in (0, \infty)$ and $q \in (0, \infty]$.

Now we are interested in the relation of $\mathring{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ to $A_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Lemma 2.26. *Let $A \in \{B, F\}$ and $s \in \mathbb{R}$. Then*

- (i)
$$\mathring{A}_{p,q}^{s,\tau}(\mathbb{R}^n) = A_{p,q}^{s,\tau}(\mathbb{R}^n) \quad \text{if and only if} \quad \tau = 0 \quad \text{and} \quad q \in (0, \infty).$$

- (ii)
$$\mathring{\mathcal{N}}_{u,p,q}^s(\mathbb{R}^n) = \mathcal{N}_{u,p,q}^s(\mathbb{R}^n) \quad \text{if and only if} \quad q \in (0, \infty).$$

Lemma 2.26 shows the essential difference between the scales $A_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)$. Under the conditions of Theorem 2.12, we know that

$$\langle A_{p_0,q_0}^{s_0,\tau_0}(\mathbb{R}^n), A_{p_1,q_1}^{s_1,\tau_1}(\mathbb{R}^n) \rangle_{\Theta} \hookrightarrow \langle A_{p_0,q_0}^{s_0,\tau_0}(\mathbb{R}^n), A_{p_1,q_1}^{s_1,\tau_1}(\mathbb{R}^n), \Theta \rangle = A_{p,q}^{s,\tau}(\mathbb{R}^n). \quad (2.6)$$

But this can be easily improved.

Lemma 2.27. *Let $s_0 \neq s_1$.*

- (i) *Under the conditions of Theorem 2.12(i), it holds true that*

$$\mathring{A}_{p,q}^{s,\tau}(\mathbb{R}^n) \hookrightarrow \langle A_{p_0,q_0}^{s_0,\tau_0}(\mathbb{R}^n), A_{p_1,q_1}^{s_1,\tau_1}(\mathbb{R}^n) \rangle_{\Theta} \hookrightarrow \mathring{B}_{\infty,\infty}^{s+n\tau-n/p}(\mathbb{R}^n) \cap A_{p,q}^{s,\tau}(\mathbb{R}^n).$$

- (ii) *Under the conditions of Theorem 2.12(ii), it holds true that*

$$\mathring{\mathcal{N}}_{u,p,q}^s(\mathbb{R}^n) \hookrightarrow \langle \mathcal{N}_{u_0,p_0,q_0}^{s_0}(\mathbb{R}^n), \mathcal{N}_{u_1,p_1,q_1}^{s_1}(\mathbb{R}^n) \rangle_{\Theta} \hookrightarrow \mathring{B}_{\infty,\infty}^{s+n\tau-n/p}(\mathbb{R}^n) \cap \mathcal{N}_{u,p,q}^s(\mathbb{R}^n).$$

- (iii) *Let $\tau \in (0, \frac{1}{p})$. Then, under the conditions of Theorem 2.12(i), it holds true that*

$$\langle A_{p_0,q_0}^{s_0,\tau_0}(\mathbb{R}^n), A_{p_1,q_1}^{s_1,\tau_1}(\mathbb{R}^n) \rangle_{\Theta} \subsetneq A_{p,q}^{s,\tau}(\mathbb{R}^n).$$

Lemma 2.27 yields the following problem: under which conditions, we have

$$\langle A_{p_0,q_0}^{s_0,\tau_0}(\mathbb{R}^n), A_{p_1,q_1}^{s_1,\tau_1}(\mathbb{R}^n) \rangle_{\Theta} = \mathring{A}_{p,q}^{s,\tau}(\mathbb{R}^n) \quad (2.7)$$

and

$$\langle \mathcal{N}_{u_0,p_0,q_0}^{s_0}(\mathbb{R}^n), \mathcal{N}_{u_1,p_1,q_1}^{s_1}(\mathbb{R}^n) \rangle_{\Theta} = \mathring{\mathcal{N}}_{u,p,q}^s(\mathbb{R}^n), \quad (2.8)$$

respectively? The answer is not always yes, but sometimes. The situation is better understood in case $\tau_0 = \tau_1 = 0$.

Theorem 2.28. *Let $\Theta \in (0, 1)$, $s_i \in \mathbb{R}$, $\tau_i \in [0, \infty)$, $p_i, q_i \in (0, \infty]$ and $u_i \in [p_i, \infty]$, $i \in \{0, 1\}$, such that $s = (1 - \Theta)s_0 + \Theta s_1$, $\tau = (1 - \Theta)\tau_0 + \Theta\tau_1$,*

$$\frac{1}{p} = \frac{1 - \Theta}{p_0} + \frac{\Theta}{p_1}, \quad \frac{1}{q} = \frac{1 - \Theta}{q_0} + \frac{\Theta}{q_1} \quad \text{and} \quad \frac{1}{u} = \frac{1 - \Theta}{u_0} + \frac{\Theta}{u_1}.$$

Let $A \in \{B, F\}$.

- (i) *If $\min\{p_0 + q_0, p_1 + q_1\} < \infty$, then*

$$\langle A_{p_0,q_0}^{s_0}(\mathbb{R}^n), A_{p_1,q_1}^{s_1}(\mathbb{R}^n) \rangle_{\Theta} = \mathring{A}_{p,q}^s(\mathbb{R}^n) = \mathring{A}_{p,q}^s(\mathbb{R}^n) = A_{p,q}^s(\mathbb{R}^n).$$

- (ii) *If either $s_0 \neq s_1$ or $s_0 = s_1$ and $q_0 \neq q_1$, then*

$$\langle B_{\infty,q_0}^{s_0}(\mathbb{R}^n), B_{\infty,q_1}^{s_1}(\mathbb{R}^n) \rangle_{\Theta} = \mathring{B}_{\infty,q}^s(\mathbb{R}^n).$$

- (iii) *If $p_0 = p_1 = p < \infty$, $s_0 \neq s_1$ and $q_0 = q_1 = \infty$, then*

$$\langle A_{p,\infty}^{s_0}(\mathbb{R}^n), A_{p,\infty}^{s_1}(\mathbb{R}^n) \rangle_{\Theta} = \mathring{A}_{p,\infty}^s(\mathbb{R}^n) \subsetneq A_{p,\infty}^s(\mathbb{R}^n).$$

(iv) Let $0 < p_0 < p_1 \leq \infty$ and $q_0 = q_1 = \infty$. If $s_0 - n/p_0 > s_1 - n/p_1$, then

$$\langle B_{p_0, \infty}^{s_0}(\mathbb{R}^n), B_{p_1, \infty}^{s_1}(\mathbb{R}^n) \rangle_{\Theta} = \dot{B}_{p, \infty}^s(\mathbb{R}^n) = \dot{B}_{p, \infty}^s(\mathbb{R}^n) \subsetneq B_{p, \infty}^s(\mathbb{R}^n).$$

(v) Let $0 < p_0 < p_1 < \infty$ and $q_0 = q_1 = \infty$. If $s_0 - n/p_0 \geq s_1 - n/p_1$, then

$$\langle F_{p_0, \infty}^{s_0}(\mathbb{R}^n), F_{p_1, \infty}^{s_1}(\mathbb{R}^n) \rangle_{\Theta} = \dot{F}_{p, \infty}^s(\mathbb{R}^n) = \dot{F}_{p, \infty}^s(\mathbb{R}^n) \subsetneq F_{p, \infty}^s(\mathbb{R}^n).$$

(vi) Let $0 < p_0 < p_1 \leq \infty$ and $q_0 = q_1 = \infty$. If $s_0 - n/p_0 \leq s_1 - n/p_1$, then

$$\dot{B}_{p, \infty}^s(\mathbb{R}^n) = \dot{B}_{p, \infty}^s(\mathbb{R}^n) \hookrightarrow \langle B_{p_0, \infty}^{s_0}(\mathbb{R}^n), B_{p_1, \infty}^{s_1}(\mathbb{R}^n) \rangle_{\Theta} \subsetneq B_{p, \infty}^s(\mathbb{R}^n).$$

(vii) Let $0 < p_0 < p_1 = \infty$, $q_0 = \infty$ and $q_1 \in (0, \infty)$. If $s_0 - n/p_0 > s_1$, then

$$\langle B_{p_0, \infty}^{s_0}(\mathbb{R}^n), B_{\infty, q_1}^{s_1}(\mathbb{R}^n) \rangle_{\Theta} = \dot{B}_{p, q}^s(\mathbb{R}^n) = \dot{B}_{p, q}^s(\mathbb{R}^n) = B_{p, q}^s(\mathbb{R}^n).$$

The cases (ii) and (iv) through (vi) of Theorem 2.28 are representing examples for

$$\langle X_0, X_1 \rangle_{\Theta} \neq \langle X_0, X_1, \Theta \rangle;$$

see Theorem 2.12.

We supplement Theorem 2.28 by two results for $\tau \in (0, \infty)$. In the first case, we fix u and p .

Theorem 2.29. Let $0 < p \leq u \leq \infty$, $\tau \in [0, \infty)$, $s_0, s_1 \in \mathbb{R}$ and $q_0, q_1 \in (0, \infty]$. Let $s := (1 - \Theta)s_0 + \Theta s_1$ and $\frac{1}{q} := \frac{1 - \Theta}{q_0} + \frac{\Theta}{q_1}$. Assume either $s_0 \neq s_1$ or $s_0 = s_1$ and $q_0 \neq q_1$. Then

$$\langle A_{p, q_0}^{s_0, \tau}(\mathbb{R}^n), A_{p, q_1}^{s_1, \tau}(\mathbb{R}^n) \rangle_{\Theta} = \dot{A}_{p, q}^{s, \tau}(\mathbb{R}^n), \quad A \in \{B, F\}.$$

In the second case, we consider large τ .

Theorem 2.30. Suppose either $\tau_i \in (1/p_i, \infty)$ and $q_i \in (0, \infty]$ or $\tau_i = 1/p_i > 0$ and $q_i = \infty$, $i \in \{0, 1\}$. If $s_0 \neq s_1$, then

$$\langle A_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), A_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n) \rangle_{\Theta} = \dot{A}_{p, q}^{s, \tau}(\mathbb{R}^n)$$

for any pair $A, \mathcal{A} \in \{B, F\}$.

It is still an open problem whether Theorems 2.29 and 2.30 can be extended to a greater range of parameters or not. Definitely it is not true for all parameter constellations; see (2.9) with $q = 2$ and compare with Lemma 2.26(i).

Remark 2.31. Again we have to mention the fundamental article of Frazier and Jawerth [26] for a number of further results. There the following formulas are proved:

$$\begin{aligned} \langle F_{p_0, q_0}^{s_0}(\mathbb{R}^n), F_{p_1, q_1}^{s_1}(\mathbb{R}^n) \rangle_{\Theta} &= F_{p, q}^s(\mathbb{R}^n), \\ \langle F_{p_0, q_0}^{s_0, 1/p_0}(\mathbb{R}^n), F_{p_1, q_1}^{s_1, 1/p_1}(\mathbb{R}^n) \rangle_{\Theta} &= F_{p, q}^{s, 1/p}(\mathbb{R}^n) = F_{\infty, q}^s(\mathbb{R}^n), \\ \langle F_{p_0, q_0}^{s_0, 1/p_0}(\mathbb{R}^n), B_{\infty, \infty}^{s_1, 0}(\mathbb{R}^n) \rangle_{\Theta} &= F_{p, q}^{s, 1/p}(\mathbb{R}^n) = F_{\infty, q}^s(\mathbb{R}^n) \end{aligned} \quad (2.9)$$

as well as

$$\langle F_{p_0, q_0}^{s_0, 0}(\mathbb{R}^n), F_{p_1, q_1}^{s_1, 1/p_1}(\mathbb{R}^n) \rangle_{\Theta} = F_{p, q}^{s, 0}(\mathbb{R}^n)$$

and

$$\begin{aligned} \langle F_{p_0, q_0}^{s_0, 0}(\mathbb{R}^n), F_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n) \rangle_{\Theta} &= \langle F_{p_0, q_0}^{s_0, 0}(\mathbb{R}^n), B_{\infty, \infty}^{s_1 + n(\tau_1 - 1/p_1)}(\mathbb{R}^n) \rangle_{\Theta} \\ &= F_{p, q}^{s + n(\tau - 1/p) + n(1 - \Theta)/p_0, 0}(\mathbb{R}^n) \end{aligned}$$

if $p_0, p_1 \in (0, \infty)$ and $\tau_1 \in (1/p_1, \infty)$ (and with no further restrictions); see [26, Corollary 8.4].

There is one more case where we can calculate the associated interpolation space.

Theorem 2.32. Suppose $0 < p \leq u < \infty$, $s_0, s_1 \in \mathbb{R}$ and $q_i \in (0, \infty)$, $i \in \{0, 1\}$. Let $s := (1 - \Theta)s_0 + \Theta s_1$ and $\frac{1}{q} := \frac{1 - \Theta}{q_0} + \frac{\Theta}{q_1}$. Then

$$\langle \mathcal{N}_{u,p,q_0}^{s_0}(\mathbb{R}^n), \mathcal{N}_{u,p,q_1}^{s_1}(\mathbb{R}^n) \rangle_{\Theta} = \dot{\mathcal{N}}_{u,p,q}^s(\mathbb{R}^n)$$

follows for all $\Theta \in (0, 1)$.

Up to now, Theorem 2.29 (resp. Theorem 2.32) is the only answer we have to the question in (2.7) (resp. (2.8)) in case $0 < \tau < 1/p$ (resp. $0 < p < u < \infty$). Let us now have a closer look onto this problem for $p_0 < p_1$ in the most simple situations of Morrey spaces itself.

The Peetre-Gagliardo method and Morrey-Campanato spaces

Now we are interested in $\langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_{\Theta}$ and its relation to $\dot{\mathcal{M}}_p^u(\mathbb{R}^n)$, $\dot{\mathcal{M}}_p^u(\mathbb{R}^n)$ and $\mathcal{M}_p^u(\mathbb{R}^n)$. There is one more space of certain interest in the framework of Morrey spaces. We define the *space* $\dot{\mathcal{M}}_p^u(\mathbb{R}^n)$ as the closure in $\mathcal{M}_p^u(\mathbb{R}^n)$ of the set of compactly supported functions. The next lemma gives explicit descriptions of $\dot{\mathcal{M}}_p^u(\mathbb{R}^n)$, $\mathcal{M}_p^u(\mathbb{R}^n)$ and $\dot{\mathcal{M}}_p^u(\mathbb{R}^n)$ very much in the spirit of the original definition of Morrey spaces. This is of interest for its own.

Lemma 2.33. Let $1 \leq p < u < \infty$. Then

(i) $\dot{\mathcal{M}}_p^u(\mathbb{R}^n)$ is equal to the collection of all $f \in \mathcal{M}_p^u(\mathbb{R}^n)$ having the following three properties:

$$\lim_{r \downarrow 0} |B(y, r)|^{1/u-1/p} \left[\int_{B(y, r)} |f(x)|^p dx \right]^{1/p} = 0, \quad (2.10)$$

$$\lim_{r \rightarrow \infty} |B(y, r)|^{1/u-1/p} \left[\int_{B(y, r)} |f(x)|^p dx \right]^{1/p} = 0, \quad (2.11)$$

both uniformly in $y \in \mathbb{R}^n$, and

$$\lim_{|y| \rightarrow \infty} |B(y, r)|^{1/u-1/p} \left[\int_{B(y, r)} |f(x)|^p dx \right]^{1/p} = 0 \quad (2.12)$$

uniformly in $r \in (0, \infty)$.

(ii) $\dot{\mathcal{M}}_p^u(\mathbb{R}^n)$ is equal to the collection of all $f \in \mathcal{M}_p^u(\mathbb{R}^n)$ such that (2.11) (uniformly in $y \in \mathbb{R}^n$) and (2.12) (uniformly in $r \in (0, \infty)$) hold true.

(iii) $\dot{\mathcal{M}}_p^u(\mathbb{R}^n)$ is equal to the collection of all $f \in \mathcal{M}_p^u(\mathbb{R}^n)$ such that (2.10) holds true uniformly in $y \in \mathbb{R}^n$.

Remark 2.34. The restriction $p \geq 1$ of Lemma 2.33 comes into play with the construction of smooth approximations. In addition, this condition is not needed for the proofs that $\dot{\mathcal{M}}_p^u(\mathbb{R}^n)$, $\mathcal{M}_p^u(\mathbb{R}^n)$ and $\dot{\mathcal{M}}_p^u(\mathbb{R}^n)$ have the claimed properties.

There are simple, but important examples of functions explaining the difference between these spaces. Let ψ be a function as in (5.1) in Appendix. For $\alpha \in (0, \infty)$ and $x \in \mathbb{R}^n \setminus \{0\}$, let

$$f_{\alpha}(x) := |x|^{-\alpha}, \quad (2.13)$$

$$g_{\alpha}(x) := \psi(x) |x|^{-\alpha}, \quad (2.14)$$

$$h_{\alpha}(x) := (1 - \psi(x)) |x|^{-\alpha}. \quad (2.15)$$

Elementary calculations yield that

$$f_{\alpha} \in \mathcal{M}_p^u(\mathbb{R}^n) \iff \alpha = \frac{n}{u} \quad \text{and} \quad \alpha < \frac{n}{p}.$$

Similarly

$$g_{\alpha} \in \mathcal{M}_p^u(\mathbb{R}^n) \iff \alpha \leq \frac{n}{u} \quad \text{and} \quad \alpha < \frac{n}{p},$$

and

$$h_\alpha \in \mathcal{M}_p^u(\mathbb{R}^n) \iff \frac{n}{u} \leq \alpha.$$

In the limiting situation, we find that there exists a positive constant $C_{(p,u)}$, depending on p and u , such that, for all $r \in (0, \infty)$,

$$|B(0, r)|^{1/u-1/p} \left[\int_{B(0,r)} |x|^{-np/u} dx \right]^{1/p} = C_{(p,u)}.$$

Hence $f_{n/u} \notin \dot{\mathcal{M}}_p^u(\mathbb{R}^n)$ and $f_{n/u} \notin \dot{\mathcal{M}}_p^u(\mathbb{R}^n)$.

Lemma 2.35. *Let $0 < p < u < \infty$.*

- (i) $\dot{\mathcal{M}}_p^u(\mathbb{R}^n)$ and $\dot{\mathcal{M}}_p^u(\mathbb{R}^n)$ are proper subspaces of $\mathcal{M}_p^u(\mathbb{R}^n)$.
- (ii) It holds true that

$$\dot{\mathcal{M}}_p^u(\mathbb{R}^n) \hookrightarrow \dot{\mathcal{M}}_p^u(\mathbb{R}^n) \cap \dot{\mathcal{M}}_p^u(\mathbb{R}^n).$$

- (iii) It holds true that

$$\dot{\mathcal{M}}_p^u(\mathbb{R}^n) \not\subset \dot{\mathcal{M}}_p^u(\mathbb{R}^n) \quad \text{and} \quad \dot{\mathcal{M}}_p^u(\mathbb{R}^n) \not\subset \dot{\mathcal{M}}_p^u(\mathbb{R}^n).$$

Remark 2.36. Let $0 < p \leq u < \infty$. Sawano and Tanaka in [80] considered another subspace of Morrey spaces, $S\dot{\mathcal{M}}_p^u(\mathbb{R}^n)$. This space $S\dot{\mathcal{M}}_p^u(\mathbb{R}^n)$ is defined to be the collection of all $f \in \mathcal{M}_p^u(\mathbb{R}^n)$ which can be approximated by finite sums of multiples of characteristic functions of sets with finite Lebesgue measures in \mathbb{R}^n . It was proved in [80] that $S\dot{\mathcal{M}}_p^u(\mathbb{R}^n)$ is a proper subspace of $\mathcal{M}_p^u(\mathbb{R}^n)$ whenever $1 < p < u < \infty$.

Obviously, when $p = u \in (0, \infty)$, $\dot{\mathcal{M}}_p^u(\mathbb{R}^n)$ coincides with $S\dot{\mathcal{M}}_p^u(\mathbb{R}^n)$. For the case that $p < u$, we have the following embeddings:

$$\left(\dot{\mathcal{M}}_p^u(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \right) \subset \left(S\dot{\mathcal{M}}_p^u(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \right), \quad 0 < p < u < \infty \quad (2.16)$$

and

$$S\dot{\mathcal{M}}_p^u(\mathbb{R}^n) \subset \dot{\mathcal{M}}_p^u(\mathbb{R}^n), \quad 1 \leq p < u < \infty. \quad (2.17)$$

To see (2.16), let $f \in L^\infty(\mathbb{R}^n)$ be a compactly supported function in $\mathcal{M}_p^u(\mathbb{R}^n)$ with $\text{supp } f \subset K$, where K is a compact set. Then it is well known that there exists a sequence $\{g_k\}_{k \in \mathbb{N}}$ of simple functions (i. e., a complex function whose range consists of only finitely many points) such that $\|f - g_k\|_{L^\infty(\mathbb{R}^n)} \rightarrow 0$ as $k \rightarrow \infty$. Define $f_k := g_k \chi_K$ for all $k \in \mathbb{N}$. Then each f_k is a finite sum of multiples of characteristic functions of sets with finite Lebesgue measures. Since $\text{supp } f \subset K$, we further see that

$$\|f - f_k\|_{L^\infty(\mathbb{R}^n)} = \|(f - g_k)\chi_K\|_{L^\infty(\mathbb{R}^n)} \leq \|f - g_k\|_{L^\infty(\mathbb{R}^n)} \rightarrow 0$$

as $k \rightarrow \infty$, which, together with $0 < p < u < \infty$, implies that

$$\begin{aligned} \|f - f_k\|_{\mathcal{M}_p^u(\mathbb{R}^n)} &\leq \|f - g_k\|_{L^\infty(\mathbb{R}^n)} \|\chi_K\|_{\mathcal{M}_p^u(\mathbb{R}^n)} \\ &= \|f - g_k\|_{L^\infty(\mathbb{R}^n)} \sup_{\text{balls } B} |B|^{\frac{1}{u}} \left(\frac{|B \cap K|}{|B|} \right)^{\frac{1}{p}} \\ &\lesssim \|f - g_k\|_{L^\infty(\mathbb{R}^n)} |K|^{\frac{1}{u}} \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$. This proves the above embedding (2.16).

To show (2.17), by Lemma 2.33(ii), it suffices to show that, if $f = \chi_E$ with $|E| < \infty$, then f satisfies (2.11) and (2.12). Indeed, notice that, in this case,

$$|B(y, r)|^{1/u-1/p} \left[\int_{B(y,r)} |f(x)|^p dx \right]^{1/p} = |B(y, r)|^{1/u} \frac{|E \cap B(y, r)|^{1/p}}{|B(y, r)|^{1/p}}. \quad (2.18)$$

Thus, by $p < u$ and $|E| < \infty$, we see that

$$\lim_{r \rightarrow \infty} |B(y, r)|^{1/u-1/p} \left[\int_{B(y, r)} |f(x)|^p dx \right]^{1/p} \leq w_n^{1/u-1/p} |E|^{1/p} \lim_{r \rightarrow \infty} r^{n(1/u-1/p)} = 0,$$

where w_n denotes the volume of the unit sphere. This shows that f satisfies (2.11). Next we show that f satisfies (2.12). By the above proved conclusion, we know that, for any $\varepsilon \in (0, 1)$, there exists $R_\varepsilon \in (0, \infty)$ such that, if $r > R_\varepsilon$, then, for all $y \in \mathbb{R}^n$,

$$|B(y, r)|^{1/u-1/p} \left[\int_{B(y, r)} |f(x)|^p dx \right]^{1/p} < \varepsilon.$$

On the other hand, by (2.18), there exists $r_\varepsilon := w_n^{-1/u} \varepsilon^{u/n} > 0$ such that, if $r < r_\varepsilon$, then, for all $y \in \mathbb{R}^n$,

$$|B(y, r)|^{1/u-1/p} \left[\int_{B(y, r)} |f(x)|^p dx \right]^{1/p} \leq |B(y, r)|^{1/u} < \varepsilon.$$

It remains to consider the case $r_\varepsilon \leq r \leq R_\varepsilon$. Since $f \in L^p(\mathbb{R}^n)$, it follows that there exists $L_\varepsilon \in (0, \infty)$ such that

$$\int_{\mathbb{R}^n \setminus B(0, L_\varepsilon)} |f(x)|^p dx < w_n^{1-p/u} r_\varepsilon^{n(1-u/p)} \varepsilon^p.$$

Thus, if $|y| > L_\varepsilon + R_\varepsilon$, we then have

$$B(y, r) \subset B(y, R_\varepsilon) \subset \mathbb{R}^n \setminus B(0, L_\varepsilon)$$

and hence

$$|B(y, r)|^{\frac{1}{u}-\frac{1}{p}} \left[\int_{B(y, r)} |f(x)|^p dx \right]^{\frac{1}{p}} \leq w_n^{\frac{1}{u}-\frac{1}{p}} r_\varepsilon^{n(\frac{1}{u}-\frac{1}{p})} \left[\int_{\mathbb{R}^n \setminus B(0, L_\varepsilon)} |f(x)|^p dx \right]^{\frac{1}{p}} < \varepsilon.$$

Combining the above estimates, we then know that f satisfies (2.12). This shows that, if $1 \leq p < u < \infty$, then $S\mathcal{M}_p^u(\mathbb{R}^n) \subset \mathcal{M}_p^u(\mathbb{R}^n)$ and hence proves (2.17).

In view of Proposition 2.21, we need to study intersections of Morrey spaces.

Lemma 2.37. *Let $0 < p_0 \leq u_0 < \infty$ and $0 < p_1 \leq u_1 < \infty$ such that $p_0 \leq p_1$. Let $\Theta \in (0, 1)$,*

$$\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \quad \text{and} \quad \frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}.$$

Assume that $u_1 > u$ and $p < u$. Then $\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$ is not dense in $\mathcal{M}_p^u(\mathbb{R}^n)$. If, in addition, $p \in [1, \infty)$, then

$$\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \hookrightarrow \mathring{\mathcal{M}}_p^u(\mathbb{R}^n).$$

Corollary 2.38. *Let $0 < p_i \leq u_i < \infty$, $i \in \{0, 1\}$, and $\Theta \in (0, 1)$. Let*

$$\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \quad \text{and} \quad \frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}.$$

(i) *It holds true that*

$$\langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_\Theta \hookrightarrow \mathcal{M}_p^u(\mathbb{R}^n)$$

and the embedding is always proper except the trivial cases consisting in

(a) $p_0 = p_1$ and $u_0 = u_1$, or

(b) $p_0 = u_0$ and $p_1 = u_1$.

(ii) In addition, assume $p_0 < p_1$ and $p_0 u_1 = p_1 u_0$. Then $\mathring{\mathcal{M}}_p^u(\mathbb{R}^n)$ is a proper subspace of

$$\langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_{\Theta}.$$

(iii) Assume $p_0 < p_1$, $p_0 u_1 = p_1 u_0$ and $p \in [1, \infty)$. Then

$$\langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_{\Theta} \hookrightarrow \mathring{\mathcal{M}}_p^u(\mathbb{R}^n)$$

and this embedding is proper.

By Corollary 2.38(iii), we know that we have to introduce new spaces to obtain an explicit description of $\langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_{\Theta}$.

Definition 2.39. Let $0 < p_i < u_i < \infty$, $i \in \{0, 1\}$, $p_0 \leq p_1$ and $\Theta \in (0, 1)$. Define

$$\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \quad \text{and} \quad \frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}.$$

Then the space $\mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n)$ is defined as the collection of all $f \in L_{p_1}^{\text{loc}}(\mathbb{R}^n)$ such that

$$I_1(f) := \sup_{y \in \mathbb{R}^n} \sup_{0 < r < 1} |B(y, r)|^{1/u-1/p} \left[\int_{B(y, r)} |f(x)|^p dx \right]^{1/p} < \infty, \quad (2.19)$$

$$\lim_{r \downarrow 0} |B(y, r)|^{1/u-1/p} \left[\int_{B(y, r)} |f(x)|^p dx \right]^{1/p} = 0 \quad (2.20)$$

uniformly in $y \in \mathbb{R}^n$,

$$I_2(f) := \sup_{y \in \mathbb{R}^n} \sup_{r \geq 1} |B(y, r)|^{\frac{1}{u_0} - \frac{1}{p_0}} \left[\int_{B(y, r)} |f(x)|^{p_0} dx \right]^{1/p_0} < \infty \quad (2.21)$$

and

$$I_3(f) := \sup_{y \in \mathbb{R}^n} \sup_{r \geq 1} |B(y, r)|^{\frac{1}{u_1} - \frac{1}{p_1}} \left[\int_{B(y, r)} |f(x)|^{p_1} dx \right]^{1/p_1} < \infty. \quad (2.22)$$

Define

$$\|f\|_{\mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n)} := I_1(f) + I_2(f) + I_3(f).$$

By means of these new spaces $\mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n)$, we obtain now the first main result with respect to the Peetre-Gagliardo method applied to Morrey spaces.

Theorem 2.40. Let $\Theta \in (0, 1)$, $0 < p_i < u_i < \infty$, $i \in \{0, 1\}$, $1 \leq p_0 < p_1$ and $p_0 u_1 = p_1 u_0$. Define

$$\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \quad \text{and} \quad \frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}.$$

Then

$$\langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_{\Theta} = \mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n).$$

Remark 2.41. (i) Notice that, by Corollary 2.38, we know that

$$\mathring{\mathcal{M}}_p^u(\mathbb{R}^n) \hookrightarrow \mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n) \hookrightarrow \mathring{\mathcal{M}}_p^u(\mathbb{R}^n)$$

and all embeddings are proper.

(ii) Based on Theorem 2.40, we expect that the description of $\langle A_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), A_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n) \rangle_{\Theta}$ in case $\tau_0 \in (0, 1/p_0)$ and $\tau_1 \in (0, 1/p_1)$ requires some new spaces. Observe that

$$\left\langle F_{p_0, 2}^{0, \frac{1}{p_0} - \tau_0}(\mathbb{R}^n), F_{p_1, 2}^{0, \frac{1}{p_1} - \tau_1}(\mathbb{R}^n) \right\rangle_{\Theta} = \langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_{\Theta} = \mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n)$$

with $p_0, p_1 \in (1, \infty)$ and $\tau_i = 1/u_i$, $i \in \{0, 1\}$, by means of the Littlewood-Paley characterization of the Morrey spaces, namely, $F_{p, 2}^{0, \frac{1}{p} - \frac{1}{u}}(\mathbb{R}^n) = \mathcal{M}_p^u(\mathbb{R}^n)$ for all $1 < p \leq u < \infty$ (see, for example, [115, Corollary 3.3]).

There is an important difference between the interpolation of Morrey spaces on unbounded domains and bounded ones. First, we need to recall the definition of Morrey spaces on domains.

Definition 2.42. Let $0 < p \leq u < \infty$ and let $\Omega \subset \mathbb{R}^n$ be bounded. Then the Morrey space $\mathcal{M}_p^u(\Omega)$ is defined as the set of all $f \in L_p^{\text{loc}}(\Omega)$ such that

$$\|f\|_{\mathcal{M}_p^u(\Omega)} := \sup_{x \in \Omega} \sup_{r \in (0, \infty)} |B(x, r) \cap \Omega|^{\frac{1}{u} - \frac{1}{p}} \left[\int_{B(x, r) \cap \Omega} |f(y)|^p dy \right]^{1/p} < \infty.$$

For easier reference, we will concentrate on $\Omega = (0, 1)^n$. We need to recall an embedding result of Dchumakaeva [21]:

$$W^m(\mathcal{M}_p^u((0, 1)^n)) \hookrightarrow C((0, 1)^n), \quad 1 \leq p < \infty, \quad m > \frac{n}{p}.$$

Here $W^m(\mathcal{M}_p^u((0, 1)^n))$ denotes the Sobolev space built on the Morrey space $\mathcal{M}_p^u((0, 1)^n)$ and, as usual, $C((0, 1)^n)$ denotes the space of all continuous functions on $(0, 1)^n$ equipped with the supremum norm. By means of this embedding, we can derive the following conclusion, the details being omitted.

Lemma 2.43. Let $1 \leq p \leq u < \infty$. Then

$$\begin{aligned} \left\{ f \in \mathcal{M}_p^u((0, 1)^n) : D^\alpha f \in \mathcal{M}_p^u((0, 1)^n) \text{ for all } \alpha \in \mathbb{Z}_+ \right\} \\ = \left\{ f \in C^\infty((0, 1)^n) : D^\alpha f \in L_\infty((0, 1)^n) \text{ for all } \alpha \in \mathbb{Z}_+ \right\}. \end{aligned}$$

Based on this simple lemma, it is now possible to show that there is no need for new spaces in case of bounded domains.

Theorem 2.44. Let $\Theta \in (0, 1)$, $1 \leq p_0 \leq u_0 < \infty$ and $1 \leq p_1 \leq u_1 < \infty$. If $\frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}$, $\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$ and $p_0 u_1 = p_1 u_0$, then

$$\langle \mathcal{M}_{p_0}^{u_0}((0, 1)^n), \mathcal{M}_{p_1}^{u_1}((0, 1)^n) \rangle_\Theta = \mathring{\mathcal{M}}_p^u((0, 1)^n).$$

We point out that, by a proof similar to that of Theorem 2.44, we see that the conclusion of Theorem 2.44 still holds true, if $(0, 1)^n$ is replaced by a bounded domain Ω .

The corresponding result for Besov-Morrey spaces on some domains also holds true; see Appendix, Subsection 5.5, for a definition. Here we concentrate us on Lipschitz domains $\Omega \subset \mathbb{R}^n$. By a Lipschitz domain, we mean either a special or a bounded Lipschitz domain. Recall that a *special Lipschitz domain* is an open set $\Omega \subset \mathbb{R}^n$ lying above the graph of a Lipschitz function $w : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$, namely,

$$\Omega := \{(x', x_n) \in \mathbb{R}^n : x_n > w(x')\},$$

where w satisfies that, for all $x', y' \in \mathbb{R}^{n-1}$,

$$|w(x') - w(y')| \leq A|x' - y'|$$

with a positive constant A independent of x' and y' . A *bounded Lipschitz domain* is a bounded domain $\Omega \subset \mathbb{R}^n$ whose boundary $\partial\Omega$ can be cover by a finite number of open balls B_k such that, for each k , after a suitable rotation, $\partial\Omega \cap B_k$ is a part of the graph of a Lipschitz function.

Theorem 2.45. Let $\Omega \subset \mathbb{R}^n$ be an interval if $n = 1$ or a Lipschitz domain if $n \geq 2$. Assume that $0 < p_i \leq u_i < \infty$, $s_0, s_1 \in \mathbb{R}$ and $q_i \in (0, \infty)$, $i \in \{0, 1\}$. Let $s := (1 - \Theta)s_0 + \Theta s_1$, $\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$ and $\frac{1}{q} := \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}$. If $u_0 p_1 = u_1 p_0$, then

$$\langle \mathcal{N}_{u_0, p_0, q_0}^{s_0}(\Omega), \mathcal{N}_{u_1, p_1, q_1}^{s_1}(\Omega) \rangle_\Theta = \mathring{\mathcal{N}}_{u, p, q}^s(\Omega)$$

holds true for all $\Theta \in (0, 1)$.

2.4 The complex method of interpolation for quasi-Banach spaces

The complex method in case of interpolation couples of Banach spaces is a well-studied subject; see, e. g., [7, 13, 39, 40, 94]. Here we are interested in the complex method in case of interpolation couples of certain quasi-Banach spaces.

2.4.1 Basics

We begin with some basic notation taken from [39, 40, 61]. We always assume that the quasi-Banach space X is equipped with a continuous quasi-norm $\|\cdot\|_X$ (this is always possible).

Definition 2.46. A quasi-Banach space $(X, \|\cdot\|_X)$ is said to be *analytically convex* if there is a positive constant C such that, for every polynomial $P: \mathbb{C} \rightarrow X$,

$$\|P(0)\|_X \leq C \max_{|z|=1} \|P(z)\|_X.$$

In the framework of analytically convex quasi-Banach spaces, one of the key properties of Banach spaces, the maximum modulus principle, still holds true. To recall this, let

$$S_0 := \{z \in \mathbb{C} : 0 < \Re z < 1\} \quad \text{and} \quad S := \{z \in \mathbb{C} : 0 \leq \Re z \leq 1\},$$

here and hereafter, $\Re z$ for any $z \in \mathbb{C}$ denotes the *real part* of z . The following result can be found in [39, Theorem 7.4].

Proposition 2.47. *For a quasi-Banach space $(X, \|\cdot\|_X)$, the following conditions are equivalent:*

- (i) X is analytically convex.
- (ii) There exists a positive constant C such that

$$\max\{\|f(z)\|_X : z \in S_0\} \leq C \max\{\|f(z)\|_X : z \in S \setminus S_0\}$$

for any function $f: S \rightarrow X$ which is analytic on S_0 , continuous and bounded on S .

Here f being *analytic* in the open set U means that, for given $z_0 \in U$, there exists some positive number η such that there is a power series expansion

$$f(z) = \sum_{j=0}^{\infty} x_j (z - z_0)^j, \quad x_j \in X, \quad \text{uniformly convergent for } |z - z_0| < \eta.$$

The theory of analytic functions with values in quasi-Banach spaces has been developed in [37, 38, 102]. In [37], one can find the following result.

Proposition 2.48. *Let U be an open subset of the complex plane and let X be a quasi-Banach space. Let $f_n: U \rightarrow X$ be a sequence of analytic functions. If $\lim_{n \rightarrow \infty} f_n(z) = f(z)$ uniformly on compacta, then f is also analytic.*

For quasi-Banach lattices, one knows a simple criterion for being analytically convex; see [61] and [39, Theorem 7.8].

Proposition 2.49. *For a quasi-Banach lattice $(X, \|\cdot\|_X)$ of functions, the following conditions are equivalent:*

- (i) X is analytically convex.
- (ii) X is lattice r -convex for some $r \in (0, \infty)$, i. e.,

$$\left\| \left(\sum_{j=1}^m |f_j|^r \right)^{1/r} \right\|_X \leq \left(\sum_{j=1}^m \|f_j\|_X^r \right)^{1/r}$$

for any finite family $\{f_j\}_{j=1}^m$ of functions from X .

Based on the notion of the analytical convexity, the following definition makes sense.

Definition 2.50. Let (X_0, X_1) be an interpolation couple of quasi-Banach spaces, i. e., X_0 and X_1 are continuously embedded into a larger topological vector space Y . In addition, let $X_0 + X_1$ be analytically convex. Let $\Theta \in (0, 1)$.

(i) Let $\mathcal{A} := \mathcal{A}(X_0, X_1)$ be the set of all bounded and analytic functions $f : S_0 \rightarrow X_0 + X_1$, which extend continuously to the closure S of the strip S_0 such that the traces $t \mapsto f(j + it)$ are bounded continuous functions into X_j , $j \in \{0, 1\}$. We endow \mathcal{A} with the quasi-norm

$$\|f\|_{\mathcal{A}} := \max \left\{ \sup_{t \in \mathbb{R}} \|f(it)\|_{X_0}, \sup_{t \in \mathbb{R}} \|f(1 + it)\|_{X_1} \right\}.$$

The *complex interpolation space* $[X_0, X_1]_{\Theta}$ is defined as the set of all $x \in \mathcal{A}(\Theta) := \{f(\Theta) : f \in \mathcal{A}\}$ and, for all $x \in \mathcal{A}(\Theta)$, let

$$\|x\|_{[X_0, X_1]_{\Theta}} := \inf \left\{ \|f\|_{\mathcal{A}} : f \in \mathcal{A}, f(\Theta) = x \right\}.$$

(ii) Let $\mathcal{A}_0 := \mathcal{A}_0(X_0, X_1)$ be the closure of all functions $f \in \mathcal{A}$ such that $f(z) \in X_0 \cap X_1$ for all $z \in S_0$. Then the *inner complex interpolation space* $[X_0, X_1]_{\Theta}^i$ is defined in the same way as $[X_0, X_1]_{\Theta}$ with \mathcal{A} replaced by \mathcal{A}_0 .

Remark 2.51. (i) There are different definitions of the complex method in case of quasi-Banach spaces in the literature, e. g., in Kalton et al. [39], the condition

$$X_0 \cap X_1 \quad \text{is dense in} \quad X_j, \quad j \in \{0, 1\},$$

is added to the above restrictions. We found this condition inconvenient in the cases we want to apply the complex method to smoothness spaces built on Morrey spaces. So we have tried to avoid it.

(ii) Any Banach space is analytically convex. Hence, if (X_0, X_1) is an interpolation couple of Banach spaces, the interpolation space $[X_0, X_1]_{\Theta}$ for $\Theta \in (0, 1)$ in Definition 2.50 reduces to the standard definition of $[X_0, X_1]_{\Theta}$; see Calderón [13]. This requires an additional comment, for which we follow [57, p. 49]. Calderón [13] (see also [7, 4.1]) in addition assumed

$$\lim_{|t| \rightarrow \infty} f(it) = \lim_{|t| \rightarrow \infty} f(1 + it) = 0.$$

We denote the subclass of analytic functions $f \in \mathcal{A}(X_0, X_1)$ with this additional property by $\tilde{\mathcal{A}}(X_0, X_1)$. Let $f \in \mathcal{A}(X_0, X_1)$ and $\Theta \in (0, 1)$. Then $f_{\delta}(z) := e^{\delta(z-\Theta)^2} f(z)$ for all $z \in S_0$ belongs to $\mathcal{A}(X_0, X_1)$ as well and

$$\|f_{\delta}\|_{\mathcal{A}} \leq \max \left\{ e^{\delta\Theta^2}, e^{\delta(1-\Theta)^2} \right\} \|f\|_{\mathcal{A}}.$$

Furthermore, letting $\delta \rightarrow 0$, we obtain

$$\inf_{f \in \mathcal{A}(X_0, X_1), f(\Theta)=x} \|f\|_{\mathcal{A}} = \inf_{f \in \tilde{\mathcal{A}}(X_0, X_1), f(\Theta)=x} \|f\|_{\mathcal{A}}.$$

Hence, restricting the set of admissible functions f to the subset $\tilde{\mathcal{A}}(X_0, X_1)$ is changing neither the space $[X_0, X_1]_{\Theta}$ nor the quasi-norm.

(iii) Let (X_0, X_1) be an interpolation couple of Banach spaces. Then it is well known that

$$[X_0, X_1]_{\Theta} = [X_0, X_1]_{\Theta}^i;$$

see, e. g., [94, 1.9.2]. However, it is unclear whether this is still true for general quasi-Banach cases or not.

The next three properties of $[X_0, X_1]_{\Theta}$ are essentially taken from [39, 40].

Proposition 2.52. Let (X_0, X_1) be an interpolation couple of quasi-Banach spaces.

(i) Then $\mathcal{A}(X_0, X_1)$ is a quasi-Banach space continuously embedded into $C_b(S, X_0 + X_1)$, the set of all bounded continuous functions from S to $X_0 + X_1$.

(ii) Also $[X_0, X_1]_{\Theta}$ is a quasi-Banach space.

Recall that an interpolation functor is said to be of exponent $\Theta \in (0, 1)$ if there exists a positive constant C such that, for all admissible interpolation couples of quasi-Banach spaces, (X_0, X_1) and (Y_0, Y_1) , the inequality

$$\|T\|_{F(X_0, X_1) \rightarrow F(Y_0, Y_1)} \leq C \|T\|_{X_0 \rightarrow Y_0}^{1-\Theta} \|T\|_{X_1 \rightarrow Y_1}^{\Theta}$$

holds true for all linear operators $T \in \mathcal{L}(X_0, Y_0) \cap \mathcal{L}(X_1, Y_1)$.

Proposition 2.53. *Let (X_0, X_1) and (Y_0, Y_1) be interpolation couples of quasi-Banach spaces. Assume that $X_0 + X_1$ and $Y_0 + Y_1$ are analytically convex. Let $T : X_j \rightarrow Y_j$, $j \in \{0, 1\}$, be a linear and bounded operator. Then, for each $\Theta \in (0, 1)$, T is a linear and bounded operator which maps $[X_0, X_1]_{\Theta}$ into $[Y_0, Y_1]_{\Theta}$. In addition,*

$$\|T\|_{[X_0, X_1]_{\Theta} \rightarrow [Y_0, Y_1]_{\Theta}} \leq \|T\|_{X_0 \rightarrow Y_0}^{\Theta} \|T\|_{X_1 \rightarrow Y_1}^{1-\Theta},$$

i. e., the complex method represents an exact interpolation functor of exponent Θ also in the framework of quasi-Banach spaces.

We also have the following conclusion on the retraction and the coretraction.

Proposition 2.54. *Let (X_0, X_1) and (Y_0, Y_1) be two interpolation couples of quasi-Banach spaces such that Y_j is a retract of X_j , $j \in \{0, 1\}$. Assume that $X_0 + X_1$ and $Y_0 + Y_1$ are analytically convex. Then, for each $\Theta \in (0, 1)$,*

$$[Y_0, Y_1]_{\Theta} = R([X_0, X_1]_{\Theta}).$$

We mention that the method of the retraction and the coretraction in interpolation theory can be found in many places. We refer the reader to, e. g., [7, 6.4] and [94, 1.2.4, 2.4.1, 2.4.2].

Remark 2.55. For later use, we mention that Propositions 2.52, 2.53 and 2.54 remain true for the inner complex method.

2.4.2 Complex interpolation of Morrey-Campanato and related spaces. I

In this subsection, we deal with consequences of Subsection 2.1 for the complex interpolation of Morrey-Campanato and related spaces.

First we quote a result of Yang et al. [111] (with forerunners in case $\tau = 0$ in Mendez and Mitrea [61], Kalton et al. [39, Proposition 7.7], and Sickel et al. [87]).

Lemma 2.56 ([111]). *Let $q \in (0, \infty]$, $s \in \mathbb{R}$ and $\tau \in [0, \infty)$.*

- (i) *Let $p \in (0, \infty)$. Then $F_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $f_{p,q}^{s,\tau}(\mathbb{R}^n)$ are analytically convex.*
- (ii) *Let $p \in (0, \infty]$. Then $B_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $b_{p,q}^{s,\tau}(\mathbb{R}^n)$ are analytically convex.*
- (iii) *Let $0 < p \leq u \leq \infty$. Then $\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)$ and $n_{u,p,q}^s(\mathbb{R}^n)$ are analytically convex.*

Lemma 2.57. *Let $s \in \mathbb{R}$, $p, q \in (0, \infty]$ ($p < \infty$ for F -spaces), $u \in [p, \infty]$ and $\tau \in [0, \infty)$.*

- (i) *It holds true that $\mathring{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$, $\mathring{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\mathring{\mathcal{N}}_{u,p,q}^s(\mathbb{R}^n)$ are analytically convex.*
- (ii) *It holds true that the spaces $A_{p,q}^{s,\tau}(\mathbb{R}^n)$ when $\tau \in (0, \infty)$, and $\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)$ when $p < u$, are nonseparable.*
- (iii) *If $\tau \in (0, \infty)$, then $\mathring{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is a proper subspace of $A_{p,q}^{s,\tau}(\mathbb{R}^n)$, $A \in \{B, F\}$.*
- (iv) *If $0 < p < u \leq \infty$, then $\mathring{\mathcal{N}}_{u,p,q}^s(\mathbb{R}^n)$ is a proper subspace of $\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)$.*

Whereas part (i) of Lemma 2.57 is an immediate consequence of Lemma 2.56, the remainder of Lemma 2.57 is a little bit more complicated to prove, we refer the reader to [111] for details.

It is well known that there are nice connections between the complex interpolation spaces and the corresponding Calderón products as follows (see the original article of Calderón [13], [40, Theorem 3.4] or [39, Theorem 7.9]).

Proposition 2.58. Let $(\mathfrak{X}, \mathcal{S}, \mu)$ be a complete separable metric space, μ a σ -finite Borel measure on \mathfrak{X} , and X_0, X_1 a pair of quasi-Banach lattices of functions on (\mathfrak{X}, μ) . If both X_0 and X_1 are analytically convex and separable, then $X_0 + X_1$ is also analytically convex and

$$[X_0, X_1]_\Theta = [X_0, X_1]_\Theta^i = X_0^{1-\Theta} X_1^\Theta, \quad \Theta \in (0, 1).$$

Because of the separability conditions in Proposition 2.58, we can not apply this proposition to Morrey-type spaces $A_{p,q}^{s,\tau}(\mathbb{R}^n)$ and to Besov-Morrey spaces $\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)$. However, it can be applied to subspaces obtained as the closure of the test functions. Before we turn to results of such a type, we comment on sequence spaces. For sequence spaces, one knows a little bit more. We quote Remark in front of [39, Theorem 7.10]; see also [61].

Lemma 2.59. Let X_0, X_1 be a pair of quasi-Banach sequence lattices. If both X_0 and X_1 are analytically convex and at least one is separable, then $X_0 + X_1$ is also analytically convex and

$$[X_0, X_1]_\Theta = [X_0, X_1]_\Theta^i = X_0^{1-\Theta} X_1^\Theta, \quad \Theta \in (0, 1).$$

Remark 2.60. We need to go back to the problem described in Remark 2.51. Neither in [40] nor in [39], the additional condition that $X_0 \cap X_1$ is dense in X_j , $j \in \{0, 1\}$, is used in the proofs of [40, Theorem 3.4] and [39, Theorem 7.9]. Hence, we can avoid the use of this condition in Proposition 2.58 and Lemma 2.59.

Essentially as a consequence of Propositions 2.7 and 2.8, the wavelet characterization of the spaces under consideration (see Propositions 5.8 and 5.11 in Appendix), Proposition 2.58 and Lemma 2.59, one obtain the following result (see [111] for all details).

Proposition 2.61. Let all parameters be as in Theorem 2.12.

(i) If $\tau_0 p_0 = \tau_1 p_1$, then

$$\begin{aligned} \dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n) &= [\dot{A}_{p_0,q_0}^{s_0,\tau_0}(\mathbb{R}^n), \dot{A}_{p_1,q_1}^{s_1,\tau_1}(\mathbb{R}^n)]_\Theta = [\dot{A}_{p_0,q_0}^{s_0,\tau_0}(\mathbb{R}^n), \dot{A}_{p_1,q_1}^{s_1,\tau_1}(\mathbb{R}^n)]_\Theta \\ &= [\dot{A}_{p_0,q_0}^{s_0,\tau_0}(\mathbb{R}^n), A_{p_1,q_1}^{s_1,\tau_1}(\mathbb{R}^n)]_\Theta. \end{aligned}$$

(ii) If $p_0 u_1 = p_1 u_0$, then

$$\begin{aligned} \dot{\mathcal{N}}_{u,p,q}^s(\mathbb{R}^n) &= [\dot{\mathcal{N}}_{u_0,p_0,q_0}^{s_0}(\mathbb{R}^n), \dot{\mathcal{N}}_{u_1,p_1,q_1}^{s_1}(\mathbb{R}^n)]_\Theta = [\mathcal{N}_{u_0,p_0,q_0}^{s_0}(\mathbb{R}^n), \dot{\mathcal{N}}_{u_1,p_1,q_1}^{s_1}(\mathbb{R}^n)]_\Theta \\ &= [\dot{\mathcal{N}}_{u_0,p_0,q_0}^{s_0}(\mathbb{R}^n), \mathcal{N}_{u_1,p_1,q_1}^{s_1}(\mathbb{R}^n)]_\Theta. \end{aligned}$$

Remark 2.62. Proposition 2.61 covers almost all cases for which the complex interpolation of Besov and Triebel-Lizorkin spaces is known. In particular, we obtain the formulas

$$B_{p,q}^s(\mathbb{R}^n) = [B_{p_0,q_0}^{s_0}(\mathbb{R}^n), B_{p_1,q_1}^{s_1}(\mathbb{R}^n)]_\Theta \quad \text{with} \quad \max\{p_0, q_0\} < \infty,$$

and

$$F_{p,q}^s(\mathbb{R}^n) = [F_{p_0,q_0}^{s_0}(\mathbb{R}^n), F_{p_1,q_1}^{s_1}(\mathbb{R}^n)]_\Theta \quad \text{with} \quad \max\{p_0, p_1\} + \min\{q_0, q_1\} < \infty,$$

where

$$s = (1 - \Theta) s_0 + \Theta s_1 \quad \frac{1}{q} := \frac{1 - \Theta}{q_0} + \frac{\Theta}{q_1} \quad \text{and} \quad \frac{1}{p} := \frac{1 - \Theta}{p_0} + \frac{\Theta}{p_1}.$$

Here we have used the fact that

$$\dot{A}_{p,q}^s(\mathbb{R}^n) = A_{p,q}^s(\mathbb{R}^n) \iff \max\{p, q\} < \infty, \quad A \in \{B, F\};$$

see [96, Theorem 2.3.3]. These interpolation formulas have been known before, we refer the reader to Bergh, Löfström [7, Theorem 6.4.5], Triebel [94, 2.4.1/2], [95], Frazier, Jawerth [26], Mendez, Mitrea [61] and Kalton et al. [39]. In the next subsection, we shall continue this discussion by considering those situations where both spaces are non-separable.

2.4.3 Complex interpolation of Morrey-Campanato and related spaces. II

Notice that, in Proposition 2.61, at least one of the interpolated spaces should be the closure of test functions. The natural question here is, can we remove this restriction and calculate $[A_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), A_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)]_\Theta$ and/or $[\mathcal{N}_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n), \mathcal{N}_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n)]_\Theta$? In such a situation, we can not use Proposition 2.58 because of the non-separability of the spaces involved. Instead one can apply an argument of Shestakov [82, 83].

Proposition 2.63. *Let (X_0, X_1) be a couple of Banach lattices and $\Theta \in (0, 1)$. Then*

$$[X_0, X_1]_\Theta = [X_0, X_1, X_0^{1-\Theta} X_1^\Theta, \#] .$$

Using the inner complex method instead of the usual complex method, the following generalization of Proposition 2.63 was obtained in [112].

Proposition 2.64. *Let (X_0, X_1) be a couple of analytically convex quasi-Banach lattices and $\Theta \in (0, 1)$. Then*

$$[X_0, X_1]_\Theta^i = [X_0, X_1, X_0^{1-\Theta} X_1^\Theta, \#] .$$

In view of Proposition 2.20, we have the following obvious conclusion which will be our main tool in this subsection.

Corollary 2.65. *Let (X_0, X_1) be a couple of analytically convex quasi-Banach lattices and $\Theta \in (0, 1)$. If*

$$X_0^{1-\Theta} X_1^\Theta = \langle X_0, X_1, \Theta \rangle ,$$

then

$$[X_0, X_1]_\Theta^i = \langle X_0, X_1 \rangle_\Theta$$

follows.

Arguing first on the level of sequence spaces and then transferring the result to function spaces by means of the method of the retraction and the coretraction, we obtain the following.

Theorem 2.66. *Theorems 2.28, 2.29, 2.30 and 2.32 remain true with $\langle \cdot, \cdot \rangle_\Theta$ replaced by $[\cdot, \cdot]_\Theta^i$.*

Complex interpolation of Besov and Triebel-Lizorkin spaces

Theorem 2.66 has some very interesting consequences for the complex interpolation of Besov and Triebel-Lizorkin spaces. On the one side, it covers all the cases discussed in Remark 2.62 (see Theorem 2.28(i)), on the other hand, it provides the most complete collection of results concerning the complex interpolation of Besov and Triebel-Lizorkin spaces where both spaces are non-separable. In particular, we have

$$\dot{B}_{p,q}^s(\mathbb{R}^n) = [B_{p_0, q_0}^{s_0}(\mathbb{R}^n), B_{p_1, q_1}^{s_1}(\mathbb{R}^n)]_\Theta^i ,$$

if the standard assumptions

$$s = (1 - \Theta) s_0 + \Theta s_1 , \quad \frac{1}{q} := \frac{1 - \Theta}{q_0} + \frac{\Theta}{q_1} \quad \text{and} \quad \frac{1}{p} := \frac{1 - \Theta}{p_0} + \frac{\Theta}{p_1} \quad (2.23)$$

are fulfilled and, in addition, one of the further sets of the following restrictions holds true:

- (a) $s_0 \neq s_1$ and $p_0 = p_1 = \infty$;
- (b) $s_0 = s_1$, $p_0 = p_1 = \infty$ and $q_0 < q_1$;
- (c) $s_0 \neq s_1$, $0 < p_0 = p_1 < \infty$ and $q_0 = q_1 = \infty$;
- (d) $0 < p_0 < p_1 < \infty$, $s_0 - n/p_0 > s_1 - n/p_1$ and $q_0 = q_1 = \infty$.

Similarly,

$$\dot{F}_{p,q}^s(\mathbb{R}^n) = [F_{p_0, q_0}^{s_0}(\mathbb{R}^n), F_{p_1, q_1}^{s_1}(\mathbb{R}^n)]_\Theta^i ,$$

if the standard assumptions (2.23) are fulfilled and, in addition, one of the further sets of the following restrictions holds true:

(e) $s_0 \neq s_1$, $0 < p_0 = p_1 < \infty$ and $q_0 = q_1 = \infty$;

(f) $0 < p_0 < p_1 < \infty$, $s_0 - n/p_0 \geq s_1 - n/p_1$ and $q_0 = q_1 = \infty$.

By a closer look onto these different sets of restrictions, we find that the space

$$[B_{p_0, q_0}^{s_0}(\mathbb{R}^n), B_{p_1, q_1}^{s_1}(\mathbb{R}^n)]_{\Theta}^i$$

is not known if either

$$0 < p_0 < p_1 \leq \infty, \quad s_0 - n/p_0 \leq s_1 - n/p_1 \quad \text{and} \quad q_0 = q_1 = \infty$$

or

$$0 < p_0 < p_1 = \infty, \quad s_0 - n/p_0 \leq s_1, \quad q_0 = \infty \quad \text{and} \quad q_1 \in (0, \infty).$$

Similarly, the space $[F_{p_0, q_0}^{s_0}(\mathbb{R}^n), F_{p_1, q_1}^{s_1}(\mathbb{R}^n)]_{\Theta}^i$ is not known if

$$0 < p_0 < p_1 < \infty, \quad s_0 - n/p_0 \leq s_1 - n/p_1 \quad \text{and} \quad q_0 = q_1 = \infty.$$

We add a comment to a formula stated in [7, Theorem 6.4.5], which claims that

$$B_{p, q}^s(\mathbb{R}^n) = [B_{p_0, q_0}^{s_0}(\mathbb{R}^n), B_{p_1, q_1}^{s_1}(\mathbb{R}^n)]_{\Theta}$$

for all $s_0, s_1 \in \mathbb{R}$ and $p_0, p_1, q_0, q_1 \in [1, \infty]$. By our previous remarks, this can not be true in this generality. There are plenty of counterexamples if $p_0 + q_0 = p_1 + q_1 = \infty$.

Remark 2.67. We shall make some comments to the literature. The formula

$$[B_{p, \infty}^{s_0}(\mathbb{R}^n), B_{p, \infty}^{s_1}(\mathbb{R}^n)]_{\Theta} = \dot{B}_{p, \infty}^s(\mathbb{R}^n) \subsetneq B_{p, \infty}^s(\mathbb{R}^n)$$

has been proved by Triebel [94, Theorem 2.4.1] under the restrictions $s_0 \neq s_1$ and $p \in (1, \infty)$. The counterparts of part (iv), (v) and (vi) of Theorem 2.28 for the complex method have been proved before in Sickel et al. [87] (see also [88]). Also, Sawano and Tanaka [79] have studied the complex interpolation of Besov-Morrey and Triebel-Lizorkin-Morrey spaces with fixed s and fixed p (Banach cases), in which they proved

$$[\mathcal{N}_{u, p, q_0}^s(\mathbb{R}^n), \mathcal{N}_{u, p, q_1}^s(\mathbb{R}^n)]_{\Theta} = \mathcal{N}_{u, p, q}^s(\mathbb{R}^n)$$

and

$$[F_{p, q_0}^{s, \tau}(\mathbb{R}^n), F_{p, q_1}^{s, \tau}(\mathbb{R}^n)]_{\Theta} = F_{p, q}^{s, \tau}(\mathbb{R}^n),$$

under the restrictions $s \in \mathbb{R}$, $1 < p \leq u < \infty$, $q_0, q_1 \in (1, \infty]$ and $\frac{1}{q} := \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}$, altogether being special cases of Theorem 2.66.

Now we turn to Morrey spaces. By Corollary 2.65, it is not a big surprise that the space

$$[\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)]_{\Theta}^i$$

behaves as $\langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_{\Theta}$; see Corollary 2.38.

Theorem 2.68. Let $\Theta \in (0, 1)$, $0 < p_0 \leq u_0 < \infty$,

$$0 < p_1 \leq u_1 < \infty, \quad \frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1} \quad \text{and} \quad \frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}.$$

(i) It holds true that

$$[\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)]_{\Theta}^i \hookrightarrow \mathcal{M}_p^u(\mathbb{R}^n)$$

and the embedding is always proper except the trivial cases consisting in

(a) $p_0 = p_1$ and $u_0 = u_1$, or

(b) $p_0 = u_0$ and $p_1 = u_1$.

(ii) Suppose, in addition, $1 \leq p_0 < p_1$, $p_0 < u_0$ and $u_0 p_1 = u_1 p_0$. Then

$$[\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)]_{\Theta}^i = \mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n).$$

Theorem 2.68(i) extends and supplements the negative result (1.5) of Lemarié-Rieusset [46, Theorem 3], already mentioned in Section 1 of this article, to the case of quasi-Banach spaces.

Concerning the description of $[A_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), A_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)]_{\Theta}^i$, $p_0 \neq p_1$, $\tau_i \in (0, 1/p_i)$, $i \in \{0, 1\}$, we have very little to say.

Proposition 2.69. Let $\Theta \in (0, 1)$, $s_i \in \mathbb{R}$, $p_i \in (0, \infty)$, $q_i \in (0, \infty]$, $\tau_i \in [0, 1/p_i)$, $i \in \{0, 1\}$, $s := (1 - \Theta)s_0 + \Theta s_1$,

$$\frac{1}{p} := \frac{1 - \Theta}{p_0} + \frac{\Theta}{p_1}, \quad \frac{1}{q} := \frac{1 - \Theta}{q_0} + \frac{\Theta}{q_1} \quad \text{and} \quad \tau := (1 - \Theta)\tau_0 + \Theta\tau_1.$$

Then

$$[F_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), F_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)]_{\Theta}^i \hookrightarrow [F_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), F_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)]_{\Theta} \hookrightarrow F_{p, q}^{s, \tau}(\mathbb{R}^n)$$

and

$$[B_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), B_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)]_{\Theta}^i \hookrightarrow [B_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), B_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)]_{\Theta} \hookrightarrow B_{p, q}^{s, \tau}(\mathbb{R}^n).$$

Remark 2.70. Also for the complex method, the article of Frazier and Jawerth [26] is an important source and contains a number of further results. There the following formulas were proved:

$$[F_{p_0, q_0}^{s_0}(\mathbb{R}^n), F_{p_1, q_1}^{s_1}(\mathbb{R}^n)]_{\Theta} = [F_{p_0, q_0}^{s_0, 0}(\mathbb{R}^n), F_{p_1, q_1}^{s_1, 0}(\mathbb{R}^n)]_{\Theta} = F_{p, q}^{s, 0}(\mathbb{R}^n) = F_{p, q}^s(\mathbb{R}^n),$$

$$[F_{p_0, q_0}^{s_0, 1/p_0}(\mathbb{R}^n), F_{p_1, q_1}^{s_1, 1/p_1}(\mathbb{R}^n)]_{\Theta} = F_{p, q}^{s, 1/p}(\mathbb{R}^n) = F_{\infty, q}^s(\mathbb{R}^n),$$

$$[F_{p_0, q_0}^{s_0, 1/p_0}(\mathbb{R}^n), B_{p_1, \infty}^{s_1, 1/p_1}(\mathbb{R}^n)]_{\Theta} = F_{p, q}^{s, 1/p}(\mathbb{R}^n) = F_{\infty, q}^s(\mathbb{R}^n)$$

as well as

$$[F_{p_0, q_0}^{s_0, 0}(\mathbb{R}^n), F_{p_1, q_1}^{s_1, 1/p_1}(\mathbb{R}^n)]_{\Theta} = F_{p, q}^{s, 0}(\mathbb{R}^n)$$

and

$$\begin{aligned} [F_{p_0, q_0}^{s_0, 0}(\mathbb{R}^n), F_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)]_{\Theta} &= [F_{p_0, q_0}^{s_0, 0}(\mathbb{R}^n), B_{\infty, \infty}^{s_1 + n(\tau_1 - 1/p_1)}(\mathbb{R}^n)]_{\Theta} \\ &= F_{p, q}^{s + n(\tau - 1/p) + n(1 - \Theta)/p_0, 0}(\mathbb{R}^n) \end{aligned}$$

if $p_0, p_1 \in [1, \infty)$, $q_0, q_1 \in [1, \infty]$, $\min\{q_0, q_1\} < \infty$, $s_0, s_1 \in \mathbb{R}$, $\tau_1 \in (1/p_1, \infty)$, $\Theta \in (0, 1)$ and s, p, q are as in Proposition 2.69 (and with no further restrictions); see [26, Corollary 8.3].

We turn to Besov-Morrey spaces. A counterpart to Proposition 2.69 holds true as follows.

Proposition 2.71. Let $\Theta \in (0, 1)$, $s_i \in \mathbb{R}$, $u_i \in (0, \infty)$, $p_i \in (0, u_i)$, $q_i \in (0, \infty]$, $i \in \{0, 1\}$,

$$s := (1 - \Theta)s_0 + \Theta s_1, \quad \frac{1}{p} := \frac{1 - \Theta}{p_0} + \frac{\Theta}{p_1}, \quad \frac{1}{u} := \frac{1 - \Theta}{u_0} + \frac{\Theta}{u_1} \quad \text{and} \quad \frac{1}{q} := \frac{1 - \Theta}{q_0} + \frac{\Theta}{q_1}.$$

Then

$$[\mathcal{N}_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n), \mathcal{N}_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n)]_{\Theta}^i \hookrightarrow [\mathcal{N}_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n), \mathcal{N}_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n)]_{\Theta} \hookrightarrow \mathcal{N}_{u, p, q}^s(\mathbb{R}^n).$$

At the end of this subsection, we consider function spaces on the unit open cube $(0, 1)^n$. Parallel to the conclusions of Theorem 2.44 for $\langle \mathcal{M}_{p_0}^{u_0}((0, 1)^n), \mathcal{M}_{p_1}^{u_1}((0, 1)^n) \rangle_{\Theta}$, and Theorem 2.45 for

$$\langle \mathcal{N}_{u_0, p_0, q_0}^{s_0}((0, 1)^n), \mathcal{N}_{u_1, p_1, q_1}^{s_1}((0, 1)^n) \rangle_{\Theta},$$

we obtain the following conclusion.

Theorem 2.72. Let $\Theta \in (0, 1)$, $1 \leq p_0 \leq u_0 < \infty$ and $1 \leq p_1 \leq u_1 < \infty$. If $\frac{1}{u} := \frac{1 - \Theta}{u_0} + \frac{\Theta}{u_1}$, $\frac{1}{p} := \frac{1 - \Theta}{p_0} + \frac{\Theta}{p_1}$ and $p_0 u_1 = p_1 u_0$, then

$$[\mathcal{M}_{p_0}^{u_0}((0, 1)^n), \mathcal{M}_{p_1}^{u_1}((0, 1)^n)]_{\Theta}^i = \hat{\mathcal{M}}_p^u((0, 1)^n).$$

2.5 The second complex method of interpolation

For the convenience of the reader, we also recall the second complex interpolation method of Calderón; see [13] or [7, 4.1].

Definition 2.73. Let (X_0, X_1) be an interpolation couple of Banach spaces, i. e., X_0 and X_1 are continuously embedded into a larger topological vector space Y . Let $\Theta \in (0, 1)$.

Let $\mathcal{G} := \mathcal{G}(X_0, X_1)$ be the set of all functions $f : S \rightarrow X_0 + X_1$ such that

- (a) $\frac{f(\cdot)}{1+|\cdot|}$ is continuous and bounded on S ;
- (b) f is analytic in S_0 ;
- (c) $f(j + it_1) - f(j + it_2)$ has values in X_j for all $(t_1, t_2) \in \mathbb{R}^2$, $j \in \{0, 1\}$;
- (d) the quantity

$$\|f\|_{\mathcal{G}} := \max \left\{ \sup_{t_1 \neq t_2} \left\| \frac{f(it_2) - f(it_1)}{t_2 - t_1} \right\|_{X_0}, \sup_{t_1 \neq t_2} \left\| \frac{f(1 + it_2) - f(1 + it_1)}{t_2 - t_1} \right\|_{X_1} \right\}$$

is finite.

The *complex interpolation space* $[X_0, X_1]^\Theta$ is defined as the set of all $x \in \mathcal{G}(\Theta) := \{f(\Theta) : f \in \mathcal{G}\}$ and, for all $x \in \mathcal{G}(\Theta)$,

$$\|x\|_{[X_0, X_1]^\Theta} := \inf \left\{ \|f\|_{\mathcal{G}} : f \in \mathcal{G}, f(\Theta) = x \right\}.$$

Some basic properties of this interpolation method are summarized in the next proposition; see [7, Theorem 4.1.4].

Proposition 2.74. Let (X_0, X_1) be an interpolation couple of Banach spaces and $\Theta \in (0, 1)$. The space $[X_0, X_1]^\Theta$ is a Banach space and the functor $[\cdot, \cdot]^\Theta$ is an exact interpolation functor of exponent Θ .

The relations of the two complex interpolation methods $[X_0, X_1]_\Theta$ and $[X_0, X_1]^\Theta$ are well understood; see [7, Theorem 4.3.1].

Proposition 2.75. Let (X_0, X_1) be an interpolation couple of Banach spaces and $\Theta \in (0, 1)$. Then

$$[X_0, X_1]_\Theta \hookrightarrow [X_0, X_1]^\Theta.$$

If, at least, one of the two spaces, X_0 and X_1 , is reflexive, then

$$[X_0, X_1]_\Theta = [X_0, X_1]^\Theta.$$

Finally, we quote the result of Lemarié-Rieusset [47], which is the reason why we recalled this interpolation method here.

Theorem 2.76. Let $\Theta \in (0, 1)$, $1 < p_0 \leq u_0 < \infty$ and $1 < p_1 \leq u_1 < \infty$. If $\frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}$, $\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$ and $p_0 u_1 = p_1 u_0$, then

$$[\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)]^\Theta = \mathcal{M}_p^u(\mathbb{R}^n).$$

Remark 2.77. (i) Theorem 2.76 shows that this second complex interpolation method has the potential to become as useful as the \pm -method in the context of Morrey and Morrey-type spaces. However, the disadvantage of this type of the complex interpolation for us consists in the limitation to Banach spaces.

(ii) In [7, pp. 90], Bergh and Löfström wrote: “We shall consider the space $[X_0, X_1]^\Theta$ more or less as a technical tool”. Probably the needs of the interpolation theory of Morrey spaces will lead to a new evaluation of this method.

2.6 The real method of interpolation

For the convenience of the reader, we recall some basic notation. First we recall Peetre's K -functional. Let (X_0, X_1) be a quasi-Banach couple. Then, for any $t \in (0, \infty)$ and any $x \in X_0 + X_1$, define

$$K(t, x, X_0, X_1) := \inf_{\substack{x = x_0 + x_1 \\ x_0 \in X_0, x_1 \in X_1}} \{ \|x_0\|_{X_0} + t \|x_1\|_{X_1} \}.$$

Definition 2.78. Let $\Theta \in (0, 1)$ and $q \in (0, \infty]$. The real interpolation space $(X_0, X_1)_{\Theta, q}$ is defined as the collection of all $x \in X_0 + X_1$ such that

$$\|x\|_{(X_0, X_1)_{\Theta, q}} := \left\{ \int_0^\infty [t^{-\Theta} K(t, x, X_0, X_1)]^q \frac{dt}{t} \right\}^{1/q} < \infty.$$

Concerning the relation between the real and the complex methods, we refer the reader, for example, to [19]. We also recall some basic properties of the real interpolation; see, e.g., [7] (Banach case) or [96, 2.4.1] (quasi-Banach case).

Proposition 2.79. Let (X_0, X_1) and (Y_0, Y_1) be any two quasi-Banach couples and $\Theta \in (0, 1)$.

- (i) It holds true that $(A_0, A_1)_{\Theta, q}$ is a quasi-Banach space, where $A \in \{X, Y\}$.
- (ii) If $T \in \mathcal{L}(X_0, Y_0) \cap \mathcal{L}(X_1, Y_1)$, then T maps $(X_0, X_1)_{\Theta, q}$ continuously into $(Y_0, Y_1)_{\Theta}$. Furthermore,

$$\|T\|_{(X_0, X_1)_{\Theta, q} \rightarrow (Y_0, Y_1)_{\Theta, q}} \leq \|T\|_{X_0 \rightarrow Y_0}^{1-\Theta} \|T\|_{X_1 \rightarrow Y_1}^{\Theta},$$

i.e., all functors $(\cdot, \cdot)_{\Theta, q}$ are exact and of exponent Θ .

2.6.1 Real interpolation with fixed p and τ (or u)

It is a well-known fact that the real interpolation of Besov and Triebel-Lizorkin spaces is helpful for fixed p . For different p , in general, Lorentz spaces instead of Lebesgue spaces come into play. This continues to be true for the spaces under consideration here. In addition, one has to fix either τ or u . Our first result concerns the real interpolation of the classes $A_{p, q}^{s, \tau}(\mathbb{R}^n)$.

Theorem 2.80. Let $\Theta \in (0, 1)$, $s_0, s_1 \in \mathbb{R}$, $s_0 < s_1$, $p \in (0, \infty)$, $\tau \in [0, 1/p)$, $q, q_0, q_1 \in (0, \infty]$ and $s := (1 - \Theta)s_0 + \Theta s_1$. Let $A, \mathcal{A} \in \{B, F\}$. Then

$$\mathcal{N}_{u, p, q}^s(\mathbb{R}^n) = (A_{p, q_0}^{s_0, \tau}(\mathbb{R}^n), \mathcal{A}_{p, q_1}^{s_1, \tau}(\mathbb{R}^n))_{\Theta, q}, \quad \frac{1}{u} := \frac{1}{p} - \tau,$$

in the sense of equivalent quasi-norms.

Remark 2.81. (i) It is a little bit surprising that the Besov-type spaces $B_{p, q}^{s, \tau}(\mathbb{R}^n)$ do not form a scale of interpolation spaces for the real method. However, in case $\tau = 0$, we get back the following well-known formulas that, for $\Theta \in (0, 1)$, $s_0, s_1 \in \mathbb{R}$, $s_0 < s_1$, $p \in (0, \infty)$, $q, q_0, q_1 \in (0, \infty]$ and $s := (1 - \Theta)s_0 + \Theta s_1$,

$$\begin{aligned} B_{p, q}^s(\mathbb{R}^n) &= \mathcal{N}_{p, p, q}^s(\mathbb{R}^n) = (B_{p, q_0}^{s_0, 0}(\mathbb{R}^n), B_{p, q_1}^{s_1, 0}(\mathbb{R}^n))_{\Theta, q} = (F_{p, q_0}^{s_0, 0}(\mathbb{R}^n), B_{p, q_1}^{s_1, 0}(\mathbb{R}^n))_{\Theta, q} \\ &= (B_{p, q_0}^{s_0, 0}(\mathbb{R}^n), F_{p, q_1}^{s_1, 0}(\mathbb{R}^n))_{\Theta, q} = (F_{p, q_0}^{s_0, 0}(\mathbb{R}^n), F_{p, q_1}^{s_1, 0}(\mathbb{R}^n))_{\Theta, q}; \end{aligned}$$

see [96, Theorem 2.4.2].

- (ii) A proof of Theorem 2.80 was given in [86].

Now we turn to the real interpolation of Besov-Morrey and Triebel-Lizorkin-Morrey spaces.

Theorem 2.82. Let $\Theta \in (0, 1)$, $s_0, s_1 \in \mathbb{R}$, $s_0 < s_1$, $0 < p \leq u \leq \infty$ and $q, q_0, q_1 \in (0, \infty]$. Let $s := (1 - \Theta)s_0 + \Theta s_1$, $A, \mathcal{A} \in \{\mathcal{E}, \mathcal{N}\}$ ($u < \infty$ if either $A = \mathcal{E}$ or $\mathcal{A} = \mathcal{E}$). Then

$$\mathcal{N}_{u, p, q}^s(\mathbb{R}^n) = (A_{u, p, q_0}^{s_0}(\mathbb{R}^n), \mathcal{A}_{u, p, q_1}^{s_1}(\mathbb{R}^n))_{\Theta, q}$$

in the sense of equivalent quasi-norms.

Remark 2.83. (i) Kozono and Yamazaki [41] already considered the real interpolation of Besov-Morrey spaces. They proved that

$$(\mathcal{N}_{u,p,q_0}^{s_0}(\mathbb{R}^n), \mathcal{N}_{u,p,q_1}^{s_1}(\mathbb{R}^n))_{\Theta,q} = \mathcal{N}_{u,p,q}^s(\mathbb{R}^n)$$

under the restrictions $\Theta \in (0, 1)$, $1 < p \leq u \leq \infty$, $s_0, s_1 \in \mathbb{R}$, $s_0 < s_1$, $s = (1 - \Theta)s_0 + \Theta s_1$ and $q, q_0, q_1 \in [1, \infty]$; see also Sawano, Tanaka [79]. This has been supplemented by Mazzucato in [60, Proposition 2.7] that, for $s \in \mathbb{R}$, $\Theta \in (0, 1)$, $1 < p \leq u < \infty$ and $q, q_0, q_1 \in [1, \infty]$,

$$(\mathcal{N}_{u,p,q_0}^s(\mathbb{R}^n), \mathcal{N}_{u,p,q_1}^s(\mathbb{R}^n))_{\Theta,q} = \mathcal{N}_{u,p,q}^s(\mathbb{R}^n)$$

if, in addition, $1/q = (1 - \Theta)/q_0 + \Theta/q_1$.

(ii) Mazzucato in [60, Proposition 4.12] proved that

$$\mathcal{N}_{u,p,q}^s(\mathbb{R}^n) = (\mathcal{E}_{u,p,q}^{s_0}(\mathbb{R}^n), \mathcal{E}_{u,p,q}^{s_1}(\mathbb{R}^n))_{\Theta,q}$$

with $\Theta \in (0, 1)$, $s_0, s_1 \in \mathbb{R}$, $s_0 < s_1$, $s = (1 - \Theta)s_0 + \Theta s_1$, $1 < p \leq u < \infty$ and $q \in [1, \infty]$.

(iii) A proof of Theorem 2.82 was also given in [86].

There is one special case of particular interest. We recall that $\mathcal{N}_{u,p,\infty}^s(\mathbb{R}^n) = B_{p,\infty}^{s,\tau}(\mathbb{R}^n)$, $\tau := \frac{1}{p} - \frac{1}{u}$ (see [115, Corollary 3.3]).

Corollary 2.84. Let $\Theta \in (0, 1)$, $s_0, s_1 \in \mathbb{R}$, $s_0 < s_1$, $p \in (0, \infty]$, $\tau \in [0, 1/p]$ and $q, q_0, q_1 \in (0, \infty]$. Let $s := (1 - \Theta)s_0 + \Theta s_1$ and $A, \mathcal{A} \in \{B, F\}$. Then

$$B_{p,\infty}^{s,\tau}(\mathbb{R}^n) = (A_{p,q_0}^{s_0,\tau}(\mathbb{R}^n), \mathcal{A}_{p,q_1}^{s_1,\tau}(\mathbb{R}^n))_{\Theta,\infty}$$

in the sense of equivalent quasi-norms.

For completeness, we also treat the case $\tau \in [1/p, \infty)$.

Corollary 2.85. Let $\Theta \in (0, 1)$, $s_0, s_1 \in \mathbb{R}$, $s_0 < s_1$, $p \in (0, \infty]$, and $q, q_0, q_1 \in (0, \infty]$. Let either $q_i \in (0, \infty)$ and $\tau_i \in (1/p, \infty)$ or $q_i = \infty$ and $\tau_i \in [1/p, \infty)$, $i \in \{0, 1\}$. Let

$$s := (1 - \Theta)s_0 + \Theta s_1, \quad \tau := (1 - \Theta)\tau_0 + \Theta \tau_1 \quad \text{and} \quad \frac{1}{p} := \frac{1 - \Theta}{p_0} + \frac{\Theta}{p_1},$$

and $A, \mathcal{A} \in \{B, F\}$. Then

$$B_{\infty,q}^{s+n\tau-n/p}(\mathbb{R}^n) = (A_{p,q_0}^{s_0,\tau_0}(\mathbb{R}^n), \mathcal{A}_{p,q_1}^{s_1,\tau_1}(\mathbb{R}^n))_{\Theta,q}$$

in the sense of equivalent quasi-norms.

2.6.2 Real interpolation with different p

We summarize some known results, due to Triebel [94, Theorem 2.4.2/1] ($1 < p_0 < p_1 < \infty$, $q \in (1, \infty]$), and Frazier, Jawerth [26, Corollary 6.7].

Proposition 2.86. Let $0 < p_0 < p_1 < \infty$, $s \in \mathbb{R}$, $q \in (0, \infty]$ and $1/p := (1 - \Theta)/p_0 + \Theta/p_1$. Then

$$F_{p,q}^s(\mathbb{R}^n) = (F_{p_0,q}^s(\mathbb{R}^n), F_{p_1,q}^s(\mathbb{R}^n))_{\Theta,p}$$

and

$$F_{p_0/(1-\Theta),q}^{s,0}(\mathbb{R}^n) = (F_{p_0,q}^{s,0}(\mathbb{R}^n), F_{p_1,q}^{s,1/p_1}(\mathbb{R}^n))_{\Theta,p}.$$

Finally we turn to the real interpolation of Morrey spaces themselves. Here we only consider embeddings.

Lemma 2.87. Let $0 < p_i \leq u_i < \infty$, $i \in \{0, 1\}$, $1/p := (1 - \Theta)/p_0 + \Theta/p_1$ and $1/u := (1 - \Theta)/u_0 + \Theta/u_1$.

(i) It always holds true that

$$(\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n))_{\Theta,p} \hookrightarrow \mathcal{M}_p^u(\mathbb{R}^n).$$

(ii) Let $\min\{p_0, p_1\} > 1$. Then

$$\mathcal{M}_p^u(\mathbb{R}^n) \hookrightarrow (\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n))_{\Theta, \infty}$$

holds true if and only if $p_0/u_0 = p_1/u_1$.

(iii) If $p_0 = p_1$, then

$$(\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n))_{\Theta, \infty} \hookrightarrow \mathcal{M}_p^u(\mathbb{R}^n). \quad (2.24)$$

The embedding in Lemma 2.87(i) with $\min\{p_0, p_1\} \geq 1$ has been known for some time, we refer the reader to Mazzucato [60], Lemarié-Rieussiet [46] and Sickel [86]. Lemma 2.87(ii) is taken from Lemarié-Rieussiet [46]. Concerning Lemma 2.87(iii), we wish to mention that, in case $\min\{p_0, p_1\} \geq 1$, Lemarié-Rieussiet [46] has proved a sharper version, in which the necessity of $p_0 = p_1$ for the validity of the embedding (2.24) has been shown. Whereas Lemma 2.87 is useful, the next statement is instructive to what concerns the limitations of the real method in our context.

Theorem 2.88. Let $1 \leq p_i \leq u_i < \infty$, $i \in \{0, 1\}$, $1/p := (1-\Theta)/p_0 + \Theta/p_1$ and $1/u := (1-\Theta)/u_0 + \Theta/u_1$. Then, for all $q \in (0, \infty]$,

$$(\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n))_{\Theta, q} \neq \mathcal{M}_p^u(\mathbb{R}^n),$$

except the trivial cases consisting in

- (a) $p_0 = p_1$ and $u_0 = u_1$, or
- (b) $p_0 = u_0$, $p_1 = u_1$ and $q = p$.

We now turn to Besov-Morrey spaces. Here we are going to use the following conclusion, whose Banach version was proved in [7, Theorem 5.6.2] (see also [94, 1.18.1, 1.18.2]).

Lemma 2.89. Assume that $s_0, s_1 \in \mathbb{R}$, $p_0, p_1 \in (0, \infty)$, $s := (1-\Theta)s_0 + \Theta s_1$ and $1/p := (1-\Theta)/p_0 + \Theta/p_1$. Let (X_0, X_1) be an interpolation couple of quasi-Banach spaces. Then

$$(\ell_{p_0}^{s_0}(X_0), \ell_{p_1}^{s_1}(X_1))_{\Theta, p} = \ell_p^s((X_0, X_1)_{\Theta, p}).$$

This lemma, applied with $X_0 := \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n)$ and $X_1 := \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$, taking into account Lemma 2.87(i) and 2.89, yields the following statement.

Proposition 2.90. Let $s_i \in \mathbb{R}$, $0 < p_i \leq u_i < \infty$, $q_i \in (0, \infty]$, $i \in \{0, 1\}$, $s := (1-\Theta)s_0 + \Theta s_1$, $1/p := (1-\Theta)/p_0 + \Theta/p_1$, $1/q := (1-\Theta)/q_0 + \Theta/q_1$ and $1/u := (1-\Theta)/u_0 + \Theta/u_1$. Then

$$(\mathcal{N}_{u_0, p_0, p_0}^{s_0}(\mathbb{R}^n), \mathcal{N}_{u_1, p_1, p_1}^{s_1}(\mathbb{R}^n))_{\Theta, p} \hookrightarrow \mathcal{N}_{u, p, p}^s(\mathbb{R}^n).$$

2.7 The interpolation property

The aim of this subsection consists in a collection of the consequences of the previously obtained results for the interpolation property of linear operators.

Before turning to these results, we would like to give a comment on the positive interpolation results obtained so far. As we have seen in Subsections 2.1 through 2.6, positive results were always connected with the restriction $p_0/u_0 = p_1/u_1$ (or $\tau_0 p_0 = \tau_1 p_1$). There is an explanation which we learned from Lemarié-Rieussiet (a personal communication with the second author) as follows. The condition $p_0/u_0 = p_1/u_1$ characterizes those pairs of Morrey spaces $(\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n))$ which are connected by a bijection. More precisely, the mapping

$$T_\delta : f \mapsto (\arg f) |f|^\delta$$

is a bijection from $\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n)$ onto $\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$ if $\delta = p_0/p_1 = u_0/u_1$. Hence, positive interpolation results were only obtained within a scale of images $\{T_\delta(\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n))\}_{\delta>0}$ of a fixed Morrey space $\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n)$. The second Morrey space $\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$ has to belong to this scale and, as a result, the interpolation space will belong to as well (for some methods, not all). Without this bijectivity, we do not know any positive results.

We continue by recalling the most prominent statement concerning the interpolation property in the framework of Morrey spaces.

Lemma 2.91. Let $\Theta \in (0, 1)$, $0 < p_0 \leq u_0 < \infty$, $0 < p_1 \leq u_1 < \infty$ and define

$$\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \quad \text{and} \quad \frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}.$$

Let X_0, X_1 be an interpolation couple of quasi-Banach spaces and F an interpolation functor of exponent Θ such that

$$F(L_{p_0}(\mathbb{R}^n), L_{p_1}(\mathbb{R}^n)) \hookrightarrow L_p(\mathbb{R}^n). \quad (2.25)$$

If T is a linear operator which is bounded from X_0 to the Morrey space $\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n)$ with norm M_0 and from X_1 to the Morrey space $\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$ with norm M_1 , then T is also bounded from $F(X_0, X_1)$ to $\mathcal{M}_p^u(\mathbb{R}^n)$ and

$$\|T\|_{F(X_0, X_1) \rightarrow \mathcal{M}_p^u(\mathbb{R}^n)} \leq c M_0^{1-\Theta} M_1^\Theta,$$

where c denotes a positive constant independent of T , M_0 and M_1 .

Remark 2.92. (i) Specializing

$$X_0 := \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \quad \text{and} \quad X_1 := \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$$

and choosing T to be the identity I , we see that

$$\|I\|_{F(\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)) \rightarrow \mathcal{M}_p^u(\mathbb{R}^n)} \leq c < \infty.$$

In other words, for any functor of exponent Θ such that (2.25) holds true, we have the continuous embedding

$$F(\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)) \hookrightarrow \mathcal{M}_p^u(\mathbb{R}^n).$$

Since $(\cdot, \cdot)_{\Theta, p}$ and $[\cdot, \cdot]_\Theta$ are functors satisfying (2.25), it follows that

$$(\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n))_{\Theta, p} \hookrightarrow \mathcal{M}_p^u(\mathbb{R}^n) \quad \text{and} \quad [\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)]_\Theta \hookrightarrow \mathcal{M}_p^u(\mathbb{R}^n)$$

under the assumptions of Lemma 2.91. Notice that $\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) + \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$ is lattice r -convex for any $r \in (0, \min\{1, p_0, p_1\}]$, since $\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n)$ is lattice r -convex with $r \in (0, \min\{1, p_0\}]$ and $\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$ is lattice r -convex with $r \in (0, \min\{1, p_1\}]$.

(ii) Lemma 2.91 is implicitly contained in Spanne [89] and Peetre [68]. An extension of this lemma to more general situations (such as Besov-type or Triebel-Lizorkin-type spaces) would be highly desirable.

Let $\Theta \in (0, 1)$, $s_i \in \mathbb{R}$, $\tau_i \in [0, \infty)$, $p_i, q_i \in (0, \infty]$ and $u_i \in [p_i, \infty]$, $i \in \{0, 1\}$, such that $s := (1 - \Theta)s_0 + \Theta s_1$, $\tau := (1 - \Theta)\tau_0 + \Theta\tau_1$,

$$\frac{1}{p} := \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}, \quad \frac{1}{q} := \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1} \quad \text{and} \quad \frac{1}{u} := \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}.$$

Let (X_0, X_1) and (Y_0, Y_1) be quasi-Banach couples. In what follows, $T : X \rightarrow Y$ means that T is a linear bounded operator from X and Y . The following interpolation properties of linear operators on smoothness function spaces built on Morrey spaces are obtained in this article:

(a) Besov-type and Triebel-Lizorkin-type spaces. In addition, we assume $\tau_0 p_0 = \tau_1 p_1$. Then, from Proposition 2.10 and Theorem 2.12, we deduce that, with $A \in \{B, F\}$,

$$\begin{aligned} T : \begin{cases} A_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n) \longrightarrow Y_0 \\ A_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n) \longrightarrow Y_1 \end{cases} &\implies T : A_{p, q}^{s, \tau}(\mathbb{R}^n) \longrightarrow \langle Y_0, Y_1, \Theta \rangle; \\ T : \begin{cases} X_0 \longrightarrow A_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n) \\ X_1 \longrightarrow A_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n) \end{cases} &\implies T : \langle X_0, X_1, \Theta \rangle \longrightarrow A_{p, q}^{s, \tau}(\mathbb{R}^n). \end{aligned}$$

- (b) Besov-type and Triebel-Lizorkin-type spaces. This time we allow $\tau_0 p_0 \neq \tau_1 p_1$ but require $\tau_i \in [0, 1/p_i)$, $i \in \{0, 1\}$. Let $X_0 + X_1$ be analytically convex. Then, by Propositions 2.69 and 2.53, and Remark 2.55, we find that, with $A \in \{B, F\}$,

$$T : \begin{cases} X_0 \longrightarrow A_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n) \\ X_1 \longrightarrow A_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n) \end{cases} \implies T : [X_0, X_1]_{\Theta}^i \longrightarrow A_{p, q}^{s, \tau}(\mathbb{R}^n).$$

- (c) Besov-Morrey and Triebel-Lizorkin-Morrey spaces. In addition, we assume $p_0 u_1 = p_1 u_0$. Then, from Proposition 2.10 and Theorem 2.12, we deduce the following, with $\mathcal{A} \in \{\mathcal{N}, \mathcal{E}\}$,

$$\begin{aligned} T : \begin{cases} \mathcal{A}_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n) \longrightarrow Y_0 \\ \mathcal{A}_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n) \longrightarrow Y_1 \end{cases} &\implies T : \mathcal{A}_{u, p, q}^s(\mathbb{R}^n) \longrightarrow \langle Y_0, Y_1, \Theta \rangle; \\ T : \begin{cases} X_0 \longrightarrow \mathcal{A}_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n) \\ X_1 \longrightarrow \mathcal{A}_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n) \end{cases} &\implies T : \langle X_0, X_1, \Theta \rangle \longrightarrow \mathcal{A}_{u, p, q}^s(\mathbb{R}^n). \end{aligned}$$

- (d) Besov-Morrey spaces. We do not require $p_0 u_1 = p_1 u_0$ in this case. Let $X_0 + X_1$ be analytically convex. Then, by Proposition 2.71, Proposition 2.53 and Remark 2.55, we know that

$$T : \begin{cases} X_0 \longrightarrow \mathcal{N}_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n) \\ X_1 \longrightarrow \mathcal{N}_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n) \end{cases} \implies T : [X_0, X_1]_{\Theta}^i \longrightarrow \mathcal{N}_{u, p, q}^s(\mathbb{R}^n).$$

- (e) Besov-Morrey spaces, Besov-type and Triebel-Lizorkin-type spaces. Suppose $0 < \tau := \frac{1}{p} - \frac{1}{u} < 1/p$. Then, from Theorem 2.80 and Proposition 2.79, it follows that, in case $s_0 \neq s_1$, it holds true that

$$T : \begin{cases} A_{p, q_0}^{s_0, \tau_0}(\mathbb{R}^n) \longrightarrow Y_0 \\ A_{p, q_1}^{s_1, \tau_1}(\mathbb{R}^n) \longrightarrow Y_1 \end{cases} \implies T : \mathcal{N}_{u, p, r}^s(\mathbb{R}^n) \longrightarrow (Y_0, Y_1)_{\Theta, r}$$

and

$$T : \begin{cases} X_0 \longrightarrow A_{p, q_0}^{s_0, \tau_0}(\mathbb{R}^n) \\ X_1 \longrightarrow A_{p, q_1}^{s_1, \tau_1}(\mathbb{R}^n) \end{cases} \implies T : (X_0, X_1)_{\Theta, r} \longrightarrow \mathcal{N}_{u, p, r}^s(\mathbb{R}^n)$$

with arbitrary $r \in (0, \infty]$ and $A, \mathcal{A} \in \{B, F\}$.

- (f) Besov-Morrey spaces, Besov-type and Triebel-Lizorkin-type spaces. Then, by Theorem 2.82 and Proposition 2.79, we know that, in case $s_0 \neq s_1$, it holds true that

$$T : \begin{cases} A_{u, p, q_0}^{s_0}(\mathbb{R}^n) \longrightarrow Y_0 \\ A_{u, p, q_1}^{s_1}(\mathbb{R}^n) \longrightarrow Y_1 \end{cases} \implies T : \mathcal{N}_{u, p, r}^s(\mathbb{R}^n) \longrightarrow (Y_0, Y_1)_{\Theta, r}$$

and

$$T : \begin{cases} X_0 \longrightarrow A_{u, p, q_0}^{s_0}(\mathbb{R}^n) \\ X_1 \longrightarrow A_{u, p, q_1}^{s_1}(\mathbb{R}^n) \end{cases} \implies T : (X_0, X_1)_{\Theta, r} \longrightarrow \mathcal{N}_{u, p, r}^s(\mathbb{R}^n)$$

with arbitrary $r \in (0, \infty]$ and $A, \mathcal{A} \in \{\mathcal{N}, \mathcal{E}\}$.

- (g) Morrey spaces. It follows, from Proposition 2.79 and Lemma 2.87, that

$$\bullet \quad T : \begin{cases} X_0 \longrightarrow \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \\ X_1 \longrightarrow \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \end{cases} \implies T : (X_0, X_1)_{\Theta, p} \longrightarrow \mathcal{M}_p^u(\mathbb{R}^n);$$

- If $p_0 = p_1 = p$, then

$$T : \begin{cases} X_0 \longrightarrow \mathcal{M}_p^{u_0}(\mathbb{R}^n) \\ X_1 \longrightarrow \mathcal{M}_p^{u_1}(\mathbb{R}^n) \end{cases} \implies T : (X_0, X_1)_{\Theta, \infty} \longrightarrow \mathcal{M}_p^u(\mathbb{R}^n);$$

- Let $X_0 + X_1$ be analytically convex. Then

$$T : \begin{cases} X_0 \longrightarrow \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \\ X_1 \longrightarrow \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \end{cases} \implies T : [X_0, X_1]_{\Theta}^i \longrightarrow \mathcal{M}_p^u(\mathbb{R}^n);$$

see Theorem 2.68, Proposition 2.53 and Remark 2.55.

3 Interpolation of local spaces

Very recently, Triebel in [100] (see also [101]) systematically introduced and studied two new scales of function spaces, $\mathcal{L}^r B_{p,q}^s(\mathbb{R}^n)$ and $\mathcal{L}^r F_{p,q}^s(\mathbb{R}^n)$, which were called *local* (or *Morreyfied*) *spaces*. The original definition of these spaces relies on the appropriate wavelet decomposition of the distributions under consideration; see [100, 1.3.1]. Later on in [116], it was proved that the local spaces $\mathcal{L}^r B_{p,q}^s(\mathbb{R}^n)$ and $\mathcal{L}^r F_{p,q}^s(\mathbb{R}^n)$ coincide with the uniform spaces of the scales $B_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $F_{p,q}^{s,\tau}(\mathbb{R}^n)$, respectively. By this reason, we skip the original definition of the local spaces here and deal with the equivalent description of these classes as localized variants of $B_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $F_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Definition 3.1. Let Ψ be a non-negative smooth function in \mathbb{R}^n with compact support such that $\Psi(0) > 0$. Let $A \in \{B, F\}$, $s \in \mathbb{R}$, $\tau \in [0, \infty)$ and $p, q \in (0, \infty]$ ($p \in (0, \infty)$ if $A = F$). The *uniform space* $A_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$ is defined as the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{A_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)} := \sup_{\ell \in \mathbb{Z}^n} \|\Psi(\cdot - \ell)f(\cdot)\|_{A_{p,q}^{s,\tau}(\mathbb{R}^n)} < \infty.$$

It was proved in [115, 116] that the spaces $A_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$ are quasi-Banach spaces independent of the choice of Ψ (in the sense of equivalent quasi-norms). Obviously, $A_{p,q}^{s,\tau}(\mathbb{R}^n) \hookrightarrow A_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$. In addition, one knows, from [116], that

$$A_{p,q}^{s,\tau}(\mathbb{R}^n) = A_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n) \iff \tau \in [1/p, \infty).$$

The main result of [116] is the following identification.

Proposition 3.2. Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$ and $p, q \in (0, \infty]$ ($p \in (0, \infty)$ if $A = F$). Then

$$A_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n) = \mathcal{L}^{n(\tau-1/p)} A_{p,q}^s(\mathbb{R}^n)$$

in the sense of equivalent quasi-norms.

Parallel to the nonlocal situation one can prove the following.

Theorem 3.3. Let Θ , s , s_0 , s_1 , τ , τ_0 , τ_1 , p , p_0 , p_1 , q , q_0 , q_1 , u , u_0 , u_1 be as in Theorem 2.12 (p , p_0 , $p_1 \in (0, \infty)$ if $A = F$). If $\tau_0 p_0 = \tau_1 p_1$, then

$$\left\langle A_{p_0,q_0,\text{unif}}^{s_0,\tau_0}(\mathbb{R}^n), A_{p_1,q_1,\text{unif}}^{s_1,\tau_1}(\mathbb{R}^n), \Theta \right\rangle = A_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n).$$

Remark 3.4. One can expect that a certain part of the theory, developed in Section 2, carries over to these local spaces, which will not be dealt with in this article. For brevity, we concentrate on the most important results here.

4 Proofs

In this section, we give proofs of the results stated in Sections 2 and 3.

4.1 Proofs of results in Subsection 2.1

Proof of Theorem 2.5(iii). Under the restrictions $1 < p_i \leq u_i < \infty$, $i \in \{0, 1\}$, and $u_0 p_1 \neq u_1 p_0$, Lemarié-Rieusset [46] constructed a family of fractal sets, K_m^β , with $\beta \in [0, n)$ and $m \in \mathbb{N}$, such that the associated family of positive linear operators $T_m : \mathcal{M}_{p_i}^{u_i}(\mathbb{R}^n) \rightarrow \mathbb{R}$, $i \in \{0, 1\}$,

$$T_m f := \int_{K_m^\beta} f(x) dx,$$

has the property

$$\sup_{m \in \mathbb{N}} \frac{\|T_m\|_{\mathcal{M}_p^u(\mathbb{R}^n) \rightarrow \mathbb{R}}}{\|T_m\|_{\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \rightarrow \mathbb{R}}^{1-\Theta} \|T_m\|_{\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rightarrow \mathbb{R}}^\Theta} = \infty \quad (4.1)$$

under some conditions on β . By Proposition 2.4, this implies that

$$[\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n)]^{1-\Theta} [\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)]^\Theta \subsetneq \mathcal{M}_p^u(\mathbb{R}^n)$$

under the extra condition $\min\{p_0, p_1\} > 1$.

We claim that the above restriction $\min\{p_0, p_1\} > 1$ can be removed by studying the mapping $f \mapsto |f|^\delta$, $\delta \in (0, \infty)$. To see this, we argue by contradiction. Our assumption consists in

$$[\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n)]^{1-\Theta} [\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)]^\Theta = \mathcal{M}_p^u(\mathbb{R}^n)$$

for some p_0, p_1 such that $\min\{p_0, p_1\} = p_0 \leq 1$. We are going to use the following observations:

- (a) A function f belongs to $\mathcal{M}_p^u(\mathbb{R}^n)$ if and only if $|f|$ belongs to $\mathcal{M}_p^u(\mathbb{R}^n)$.
- (b) If f belongs to $\mathcal{M}_p^u(\mathbb{R}^n)$, then $|f|^\delta \in \mathcal{M}_{p/\delta}^{u/\delta}(\mathbb{R}^n)$ and

$$\||f|^\delta\|_{\mathcal{M}_{p/\delta}^{u/\delta}(\mathbb{R}^n)} = \|f\|_{\mathcal{M}_p^u(\mathbb{R}^n)}.$$

Indeed, by (a) and (b), we see that $T : f \mapsto (\arg f)|f|^\delta$ is a bijection with respect to the pair $(\mathcal{M}_p^u(\mathbb{R}^n), \mathcal{M}_{p/\delta}^{u/\delta}(\mathbb{R}^n))$. Now, we choose $\delta < p_0$. Then it is easy to see that

$$T\left([\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n)]^{1-\Theta} [\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)]^\Theta\right) = [\mathcal{M}_{p_0/\delta}^{u_0/\delta}(\mathbb{R}^n)]^{1-\Theta} [\mathcal{M}_{p_1/\delta}^{u_1/\delta}(\mathbb{R}^n)]^\Theta.$$

Since $T(\mathcal{M}_p^u(\mathbb{R}^n)) = \mathcal{M}_{p/\delta}^{u/\delta}(\mathbb{R}^n)$, we obtain

$$\mathcal{M}_{p/\delta}^{u/\delta}(\mathbb{R}^n) = [\mathcal{M}_{p_0/\delta}^{u_0/\delta}(\mathbb{R}^n)]^{1-\Theta} [\mathcal{M}_{p_1/\delta}^{u_1/\delta}(\mathbb{R}^n)]^\Theta,$$

but this is in conflict with the above known result for $\min\{p_0, p_1\} > 1$, which completes the proof of Theorem 2.5(iii). \square

4.2 Proofs of results in Subsection 2.2

First, we need to recall a few more notions; see, for example, [64].

Definition 4.1. (i) Let X be a quasi-Banach lattice and $p \in [1, \infty]$. The p -convexification of X , denoted by $X^{(p)}$, is defined as follows: $x \in X^{(p)}$ if and only if $|x|^p \in X$. For all $x \in X^{(p)}$, define

$$\|x\|_{X^{(p)}} := \| |x|^p \|_X^{1/p}.$$

(ii) A quasi-Banach lattice X is said to be of type \mathfrak{E} if there exists an equivalent quasi-norm $||| \cdot |||_X$ on X such that, for some $p \in [1, \infty]$, $X^{(p)}$ is a Banach lattice in the norm $\|\cdot\|_{X^{(p)}} := ||| \cdot |||_X^{1/p}$.

Let $\delta \in (0, \min\{1, p, q\}]$. Then it is easy to see that

$$\|t\|_{[a_{p,q}^{s,\tau}(\mathbb{R}^n)]^{(1/\delta)}} = \| |t|^{1/\delta} \|_{a_{p,q}^{s,\tau}(\mathbb{R}^n)}^\delta = \|t\|_{a_{p/\delta, q/\delta}^{\delta(s+n/2)-n/2, \tau\delta}(\mathbb{R}^n)}.$$

Since $p/\delta \geq 1$ and $q/\delta \geq 1$, we know that $[a_{p,q}^{s,\tau}(\mathbb{R}^n)]^{(1/\delta)}$ is a Banach lattice and hence $a_{p,q}^{s,\tau}(\mathbb{R}^n)$ is of type \mathfrak{E} . Similarly, the space $n_{u,p,q}^s(\mathbb{R}^n)$ is also of type \mathfrak{E} .

Definition 4.2. Let X_0, X_1 be a couple of quasi-Banach spaces and let X be an intermediate space with respect to $X_0 + X_1$. Then the *Gagliardo closure* of X with respect to $X_0 + X_1$, denoted by X^\sim , is defined as the collection of all $a \in X_0 + X_1$ such that there exists a sequence $\{a_i\}_{i \in \mathbb{Z}_+} \subset X$ satisfying $a_i \rightarrow a$ as $i \rightarrow \infty$ in $X_0 + X_1$ and $\|a_i\|_X \leq \lambda$ for some $\lambda < \infty$ and all $i \in \mathbb{Z}_+$. For all $a \in X^\sim$, define $\|a\|_{X^\sim} := \inf \lambda$.

Our argument will be based on the following result of Nilsson [64, Theorem 2.1].

Proposition 4.3. *Let X_0 and X_1 be two quasi-Banach lattices of type \mathfrak{E} and $\Theta \in (0, 1)$. Then*

$$\langle X_0, X_1 \rangle_\Theta = (X_0^{1-\Theta} X_1^\Theta)^\#$$

and

$$X_0^{1-\Theta} X_1^\Theta \hookrightarrow \langle X_0, X_1, \Theta \rangle \hookrightarrow (X_0^{1-\Theta} X_1^\Theta)^\sim.$$

As a preparation, we need the following result on sequence spaces which is of interest for its own.

Theorem 4.4. *Let $\Theta, s, s_0, s_1, \tau, \tau_0, \tau_1, p, p_0, p_1, q, q_0, q_1$ be as in Theorem 2.12. If $\tau_0 p_0 = \tau_1 p_1$, then*

$$\begin{aligned} \langle a_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), a_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n) \rangle_\Theta &= \left([a_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n)]^{1-\Theta} [a_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)]^\Theta \right)^\# \\ &= (a_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), a_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n), a_{p, q}^{s, \tau}(\mathbb{R}^n), \#) \end{aligned}$$

and

$$\langle a_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), a_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n), \Theta \rangle = [a_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n)]^{1-\Theta} [a_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)]^\Theta = a_{p, q}^{s, \tau}(\mathbb{R}^n).$$

The first formula in Theorem 4.4 is a direct consequence of Propositions 2.7 and 4.3. To prove the second formula, by Proposition 2.7, it suffices to show that $[a_{p, q}^{s, \tau}(\mathbb{R}^n)]^\sim = a_{p, q}^{s, \tau}(\mathbb{R}^n)$, which is the conclusion of the following lemma.

Lemma 4.5. *Under the same assumptions as in Theorem 2.12(i), the Gagliardo closure $(a_{p, q}^{s, \tau}(\mathbb{R}^n))^\sim$ of $a_{p, q}^{s, \tau}(\mathbb{R}^n)$ with respect to $a_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n) + a_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)$ is given by $a_{p, q}^{s, \tau}(\mathbb{R}^n)$.*

Proof. Clearly, under the given conditions, $a_{p, q}^{s, \tau}(\mathbb{R}^n)$ is an intermediate space with respect to the pair $(a_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), a_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n))$; see Proposition 2.7.

Let $t \in [a_{p, q}^{s, \tau}(\mathbb{R}^n)]^\sim$. Then there exists a sequence $\{t_i\}_{i \in \mathbb{Z}_+} \subset a_{p, q}^{s, \tau}(\mathbb{R}^n)$ such that $t_i \rightarrow t$ as $i \rightarrow \infty$ in $a_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n) + a_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)$ and $\|t_i\|_{a_{p, q}^{s, \tau}(\mathbb{R}^n)} \lesssim \|t\|_{(a_{p, q}^{s, \tau}(\mathbb{R}^n))^\sim}$ for all $i \in \mathbb{Z}_+$. Therefore, there exist $t_i^0 \in a_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n)$ and $t_i^1 \in a_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)$ such that

$$t - t_i = t_i^0 + t_i^1, \quad i \in \mathbb{Z}_+,$$

and

$$\|t_i^0\|_{a_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n)} + \|t_i^1\|_{a_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)} \lesssim \|t - t_i\|_{a_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n) + a_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)} \rightarrow 0$$

as $i \rightarrow \infty$. Notice that, for all $Q \in \mathcal{Q}^*$,

$$|(t_i^0)_Q| \lesssim |Q|^{\frac{s_0}{n} + \frac{1}{2} - \frac{1}{p_0} + \tau_0} \|t_i^0\|_{a_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n)} \quad \text{and} \quad |(t_i^1)_Q| \lesssim |Q|^{\frac{s_1}{n} + \frac{1}{2} - \frac{1}{p_1} + \tau_1} \|t_i^1\|_{a_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)}.$$

We then know that $(t_i^0)_Q \rightarrow 0$ and $(t_i^1)_Q \rightarrow 0$ as $i \rightarrow \infty$. Hence $(t_i)_Q \rightarrow t_Q$ as $i \rightarrow \infty$. By the Fatou Lemma, we find that

$$\|t\|_{a_{p, q}^{s, \tau}(\mathbb{R}^n)} = \left\| \left\{ \lim_{i \rightarrow \infty} (t_i)_Q \right\}_{Q \in \mathcal{Q}^*} \right\|_{a_{p, q}^{s, \tau}(\mathbb{R}^n)} \leq \liminf_{i \rightarrow \infty} \|t_i\|_{a_{p, q}^{s, \tau}(\mathbb{R}^n)} \lesssim \|t\|_{(a_{p, q}^{s, \tau}(\mathbb{R}^n))^\sim},$$

which implies that $[a_{p, q}^{s, \tau}(\mathbb{R}^n)]^\sim \hookrightarrow a_{p, q}^{s, \tau}(\mathbb{R}^n)$. This, combined with Proposition 2.7, further shows that $[a_{p, q}^{s, \tau}(\mathbb{R}^n)]^\sim = a_{p, q}^{s, \tau}(\mathbb{R}^n)$ in the sense of equivalent quasi-norms, which completes the proof of Lemma 4.5. \square

Replacing Proposition 2.7 by Proposition 2.8, via an argument similar to the proof of Theorem 4.4, we obtain the following result on the sequence space $n_{u, p, q}^s(\mathbb{R}^n)$, the details being omitted.

Theorem 4.6. *Let Θ , s , s_0 , s_1 , p , p_0 , p_1 , q , q_0 , q_1 , u , u_0 , u_1 be as in Theorem 2.12. If $p_0 u_1 = p_1 u_0$, then*

$$\begin{aligned} \langle n_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n), n_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n) \rangle_{\Theta} &= \left([n_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n)]^{1-\Theta} [n_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n)]^{\Theta} \right)^{\#} \\ &= (n_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n), n_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n), n_{u, p, q}^s(\mathbb{R}^n), \#) \end{aligned}$$

and

$$\langle n_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n), n_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n), \Theta \rangle = [n_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n)]^{1-\Theta} [n_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n)]^{\Theta} = n_{u, p, q}^s(\mathbb{R}^n).$$

Applying Theorems 4.4 and 4.6, together with Propositions 2.63 and 2.64, we have the following conclusion, the details being omitted.

Corollary 4.7. *Let Θ , s , s_0 , s_1 , τ , τ_0 , τ_1 , p , p_0 , p_1 , q , q_0 , q_1 , u , u_0 , u_1 be as in Theorem 2.12.*

(i) *If $\tau_0 p_0 = \tau_1 p_1$, then*

$$\begin{aligned} \langle a_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), a_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n) \rangle_{\Theta} &= [a_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), a_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)]_{\Theta}^i \\ &= (a_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), a_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n), a_{p, q}^{s, \tau}(\mathbb{R}^n), \#). \end{aligned}$$

(ii) *If $\min\{p_0, p_1, q_0, q_1\} \geq 1$ and $p_0 u_1 = p_1 u_0$, then*

$$\begin{aligned} \langle n_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n), n_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n) \rangle_{\Theta} &= [n_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n), n_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n)]_{\Theta}^i \\ &= (n_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n), n_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n), n_{u, p, q}^s(\mathbb{R}^n), \#). \end{aligned}$$

Proof of Theorem 2.12

Theorem 2.12(i) follows from Theorem 4.4 in combination with Proposition 5.8. Indeed, by Proposition 5.8, we know the existence of a homeomorphism $R: A_{p, q}^{s, \tau}(\mathbb{R}^n) \rightarrow a_{p, q}^{s, \tau}(\mathbb{R}^n)$ (for all parameters s , τ , p , q). Then, for any $f \in A_{p, q}^{s, \tau}(\mathbb{R}^n)$, we obtain $R(f) \in a_{p, q}^{s, \tau}(\mathbb{R}^n)$ and

$$\|R(f)\|_{a_{p, q}^{s, \tau}(\mathbb{R}^n)} \lesssim \|f\|_{A_{p, q}^{s, \tau}(\mathbb{R}^n)}.$$

Moreover, by Theorem 4.4, there exists a sequence $\{t_i\}_{i \in \mathbb{Z}} \subset a_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n) \cap a_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)$ such that $R(f) = \sum_{i \in \mathbb{Z}} t_i$ with convergence in $a_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n) + a_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)$ and, for any finite subset $F \subset \mathbb{Z}$ and any bounded sequence $\{\varepsilon_i\}_{i \in \mathbb{Z}} \subset \mathbb{C}$,

$$\left\| \sum_{i \in F} \varepsilon_i 2^{i(j-\Theta)} t_i \right\|_{a_{p_j, q_j}^{s_j, \tau_j}(\mathbb{R}^n)} \lesssim \|f\|_{A_{p, q}^{s, \tau}(\mathbb{R}^n)} \sup_{i \in \mathbb{Z}} |\varepsilon_i|, \quad j \in \{0, 1\}.$$

Notice that $f_i := R^{-1}(t_i) \in A_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n) \cap A_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)$ for all $i \in \mathbb{Z}$ and $f = \sum_{i \in \mathbb{Z}} R^{-1}(t_i)$ in $A_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n) + A_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)$. Moreover, by Proposition 2.10, we know that

$$\left\| \sum_{i \in F} \varepsilon_i 2^{i(j-\Theta)} R^{-1}(t_i) \right\|_{A_{p_j, q_j}^{s_j, \tau_j}(\mathbb{R}^n)} \lesssim \|f\|_{A_{p, q}^{s, \tau}(\mathbb{R}^n)} \sup_{i \in \mathbb{Z}} |\varepsilon_i|, \quad j \in \{0, 1\}.$$

This implies that $A_{p, q}^{s, \tau}(\mathbb{R}^n) \hookrightarrow \langle A_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), A_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n), \Theta \rangle$. The reverse embedding

$$\langle A_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), A_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n), \Theta \rangle \hookrightarrow A_{p, q}^{s, \tau}(\mathbb{R}^n)$$

follows from a similar argument to above, the details being omitted.

Similarly, Theorem 2.12(ii) is a consequence of Theorem 4.6 in combination with Proposition 5.11, the details being omitted.

Proof of Corollary 2.14

Step 1. Proof of (i). Recall that $F_{p,2}^{0,1/p-1/u}(\mathbb{R}^n) = \mathcal{M}_p^u(\mathbb{R}^n)$ if $1 < p \leq u < \infty$ (see Mazzucato [59] and Sawano [76]). Hence, if $p_0, p_1 \in (1, \infty)$, then Corollary 2.14(i) is a direct consequence of Theorem 2.12(i) with $A = F$. Furthermore, by the proof of Theorem 2.12, we know that, in this case, the Gagliardo closure of $\mathcal{M}_p^u(\mathbb{R}^n)$ in $\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) + \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$ is $\mathcal{M}_p^u(\mathbb{R}^n)$ itself.

Let us turn to the case $\min\{p_0, p_1\} \leq 1$. We claim that, also in this case, the Gagliardo closure of $\mathcal{M}_p^u(\mathbb{R}^n)$ coincides with $\mathcal{M}_p^u(\mathbb{R}^n)$. Indeed, let f belong to the Gagliardo closure of $\mathcal{M}_p^u(\mathbb{R}^n)$ in $\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) + \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$. Since $f \in \mathcal{M}_p^u(\mathbb{R}^n)$ if and only if $|f| \in \mathcal{M}_p^u(\mathbb{R}^n)$, we only need to prove that $|f| \in \mathcal{M}_p^u(\mathbb{R}^n)$. We know that there exists a sequence $\{f_j\}_j \subset \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) + \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$ such that

$$\lim_{j \rightarrow \infty} \|f - f_j\|_{\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) + \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)} = 0 \quad \text{and} \quad \sup_{j \in \mathbb{N}} \|f_j\|_{\mathcal{M}_p^u(\mathbb{R}^n)} < \infty.$$

Since $\|f| - |f_j|\| \leq \|f - f_j\|$, we conclude that

$$\lim_{j \rightarrow \infty} \| |f| - |f_j| \|_{\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) + \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)} = 0.$$

Hence $|f|$ also belongs to the Gagliardo closure of $\mathcal{M}_p^u(\mathbb{R}^n)$ in $\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) + \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$ and it can be approximated by $\{|f_j|\}_j$ in the quasi-norm $\|\cdot\|_{\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) + \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)}$.

Choose $\delta < \min\{p_0, p_1\} \leq 1$. Notice that, for any g ,

$$\|g\|_{\mathcal{M}_p^u(\mathbb{R}^n)}^\delta = \| |g|^\delta \|_{\mathcal{M}_{p/\delta}^{u/\delta}(\mathbb{R}^n)}.$$

This yields $|f_j|^\delta \in \mathcal{M}_{p/\delta}^{u/\delta}(\mathbb{R}^n)$. Since $\||f|^\delta - |f_j|^\delta\| \leq \|f - f_j\|^\delta$, we find that

$$\begin{aligned} \||f|^\delta - |f_j|^\delta\|_{\mathcal{M}_{p_0/\delta}^{u_0/\delta}(\mathbb{R}^n) + \mathcal{M}_{p_1/\delta}^{u_1/\delta}(\mathbb{R}^n)} &\leq \| |f - f_j|^\delta \|_{\mathcal{M}_{p_0/\delta}^{u_0/\delta}(\mathbb{R}^n) + \mathcal{M}_{p_1/\delta}^{u_1/\delta}(\mathbb{R}^n)} \\ &\leq \|f - f_j\|_{\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) + \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)}^\delta \rightarrow 0 \end{aligned}$$

as $j \rightarrow \infty$. Hence, $|f|^\delta$ belongs to the Gagliardo closure of $\mathcal{M}_{p/\delta}^{u/\delta}(\mathbb{R}^n)$ in $\mathcal{M}_{p_0/\delta}^{u_0/\delta}(\mathbb{R}^n) + \mathcal{M}_{p_1/\delta}^{u_1/\delta}(\mathbb{R}^n)$. Then, by the choice of δ and the above known results for $\min\{p_0, p_1\} > 1$, we conclude that $|f|^\delta \in \mathcal{M}_{p/\delta}^{u/\delta}(\mathbb{R}^n)$ and hence $|f| \in \mathcal{M}_p^u(\mathbb{R}^n)$. This proves that the Gagliardo closure of $\mathcal{M}_p^u(\mathbb{R}^n)$ in $\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) + \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$ is a subspace of $\mathcal{M}_p^u(\mathbb{R}^n)$. Now Corollary 2.14(i) follows from Proposition 4.3 in combination with Theorem 2.5(ii).

Step 2. Proof of (ii). We may argue as in the proof of Theorem 2.5(iii). First, we need to add a comment. By using the notation as in the proof of Theorem 2.5(iii), we claim that

$$\sup_{m \in \mathbb{N}} \frac{\|T_m\|_{\mathcal{M}_p^u(\mathbb{R}^n) \rightarrow \mathbb{R}}}{\max\{\|T_m\|_{\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \rightarrow \mathbb{R}}, \|T_m\|_{\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rightarrow \mathbb{R}}\}} = \infty \quad (4.2)$$

under some conditions on $\beta \in [0, n]$ (see (4.1)). Without loss of generality, we may assume $p_0/u_0 < p_1/u_1$. Then, by choosing $\beta = n(1 - p/u)$ as Lemarié-Rieusset [46] did, we find that

$$\begin{aligned} \|T_m\|_{\mathcal{M}_p^u(\mathbb{R}^n) \rightarrow \mathbb{R}} &\geq C 2^{-m(n-\beta)(1-1/p)}, \\ \|T_m\|_{\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \rightarrow \mathbb{R}} &\leq C 2^{-m(n-\beta)(1-1/p_0)} 2^{m(\beta/p_0 + n/u_0 - n/p_0)}, \\ \|T_m\|_{\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rightarrow \mathbb{R}} &\leq C 2^{-m(n-\beta)(1-1/p_1)} \end{aligned}$$

for some positive constant C independent of m ; see [46, Section 6]. In case $p < p_1$, we have

$$\lim_{m \rightarrow \infty} \frac{\|T_m\|_{\mathcal{M}_p^u(\mathbb{R}^n) \rightarrow \mathbb{R}}}{\|T_m\|_{\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rightarrow \mathbb{R}}} = \infty; \quad (4.3)$$

whereas, in case $u < u_0$, we find that

$$\lim_{m \rightarrow \infty} \frac{\|T_m\|_{\mathcal{M}_p^u(\mathbb{R}^n) \rightarrow \mathbb{R}}}{\|T_m\|_{\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \rightarrow \mathbb{R}}} = \infty. \quad (4.4)$$

Now we distinguish our considerations into three cases.

- (a) $p_0 < p_1$. In this case, $p < p_1$ and the claim in (4.2) follows from (4.3).
- (b) $p_1 < p_0$. In this case, our assumption $p_0/u_0 < p_1/u_1$ implies $u_0 > u_1$ and hence $u_0 > u$. Thus, the claim in (4.2) follows from (4.4).
- (c) $p_0 = p_1$. In this case, our assumption $p_0/u_0 < p_1/u_1$ again implies $u_0 > u$ and we can argue as in (b) to show the claim in (4.2) holds true.

The final step of the proof is now done by applying Proposition 2.10(ii), which completes the proof of Corollary 2.14.

Proof of Corollary 2.15

As a preparation we need the following technical lemma.

Lemma 4.8. *Let $\Theta \in (0, 1)$, $0 < p_0 \leq u_0 < \infty$ and $0 < p_1 \leq u_1 < \infty$ such that*

$$\frac{1}{p} = \frac{1 - \Theta}{p_0} + \frac{\Theta}{p_1} \quad \text{and} \quad \frac{1}{u} = \frac{1 - \Theta}{u_0} + \frac{\Theta}{u_1}.$$

If $u_0 p_1 = u_1 p_0$, then

$$\left[\mathcal{M}_{p_0, \text{unif}}^{u_0}(\mathbb{R}^n) \right]^{1-\Theta} \left[\mathcal{M}_{p_1, \text{unif}}^{u_1}(\mathbb{R}^n) \right]^{\Theta} = \mathcal{M}_{p, \text{unif}}^u(\mathbb{R}^n).$$

Proof. The arguments used in the proof of [56, Proposition 2.1] carry over to this locally uniform situation, the details being omitted. \square

Now we turn to the proof of Corollary 2.15.

Proof of Corollary 2.15. We have to distinguish four cases, comparing with the discussion in Remark 2.13.

Step 1. $\min\{\tau_0, \tau_1\} = 0$. In this case, according to our restriction $\tau_0 p_0 = \tau_1 p_1$, we obtain $\tau = \tau_0 = \tau_1 = 0$. This implies

$$\mathcal{L}_p^0(\mathbb{R}^n) = \mathcal{M}_{p, \text{unif}}^p(\mathbb{R}^n) = L_{p, \text{unif}}(\mathbb{R}^n);$$

see (a) of Section 1 of this article. By the same arguments as those used in the proof of Corollary 2.14, we conclude that the Gagliardo closure of $\mathcal{M}_{p, \text{unif}}^p(\mathbb{R}^n)$ with respect to $\mathcal{M}_{p_0, \text{unif}}^{p_0}(\mathbb{R}^n) + \mathcal{M}_{p_1, \text{unif}}^{p_1}(\mathbb{R}^n)$ is just $\mathcal{M}_{p, \text{unif}}^p(\mathbb{R}^n)$ itself. Now the desired conclusion of Corollary 2.15 follows from Proposition 4.3.

Step 2. $\min\{\tau_0, \tau_1\} > 0$ and $\min\{\tau_0 - 1/p_0, \tau_1 - 1/p_1\} < 0$. In this case, our restriction $\tau_0 p_0 = \tau_1 p_1$ implies

$$\max\{\tau - 1/p, \tau_0 - 1/p_0, \tau_0 - 1/p_1\} < 0.$$

In this situation, by (a) of Section 1 of this article, we know that $\mathcal{L}_p^\tau(\mathbb{R}^n) = \mathcal{M}_{p, \text{unif}}^u(\mathbb{R}^n)$ with $1/u = 1/p - \tau$, and thus we can argue as in Step 1.

Step 3. Either $\tau_0 = 1/p_0$ or $\tau_1 = 1/p_1$. In this case, our restriction $\tau_0 p_0 = \tau_1 p_1$ implies $\tau p = 1$. Using (c) of Section 1 of this article, we find that

$$\mathcal{L}_{p_0}^{1/p_0}(\mathbb{R}^n) = \mathcal{L}_{p_1}^{1/p_1}(\mathbb{R}^n) = \mathcal{L}_p^{1/p}(\mathbb{R}^n) = \text{bmo}(\mathbb{R}^n).$$

Since $\langle X, X, \Theta \rangle = X$ for any quasi-Banach space, the desired conclusion of Corollary 2.15 follows also in this case.

Step 4. $\max\{\tau_0 - 1/p_0, \tau_1 - 1/p_1\} > 0$. In this case, by (b) of Section 1 of this article, we know that

$$\mathcal{L}_{p_0}^{\tau_0}(\mathbb{R}^n) = B_{\infty, \infty}^{n(\tau_0 - 1/p_0)}(\mathbb{R}^n), \quad \mathcal{L}_{p_1}^{\tau_1}(\mathbb{R}^n) = B_{\infty, \infty}^{n(\tau_1 - 1/p_1)}(\mathbb{R}^n), \quad \mathcal{L}_p^\tau(\mathbb{R}^n) = B_{\infty, \infty}^{n(\tau - 1/p)}(\mathbb{R}^n).$$

Applying Theorem 2.12 and Proposition 5.2(iii), we find that

$$\begin{aligned} \langle B_{\infty, \infty}^{n(\tau_0 - 1/p_0)}(\mathbb{R}^n), B_{\infty, \infty}^{n(\tau_1 - 1/p_1)}(\mathbb{R}^n), \Theta \rangle &= \langle B_{p_0, q_0}^{0, \tau_0}(\mathbb{R}^n), B_{p_1, q_1}^{0, \tau_1}(\mathbb{R}^n), \Theta \rangle \\ &= B_{p, q}^{0, \tau}(\mathbb{R}^n) = B_{\infty, \infty}^{n(\tau - 1/p)}(\mathbb{R}^n) = \mathcal{L}_p^\tau(\mathbb{R}^n), \end{aligned}$$

where $q, q_0, q_1 \in (0, \infty]$ satisfy $1/q = 1/q_0 + 1/q_1$. This finishes the proof of Corollary 2.15. \square

4.3 Proofs of results in Subsection 2.3

Proof of Proposition 2.17

First we deal with the completeness of $\langle A_0, A_1 \rangle_\Theta$ in (i). Let $\{a^\ell\}_{\ell \in \mathbb{N}}$ be a fundamental sequence in $\langle A_0, A_1 \rangle_\Theta$. Then, for any $\varepsilon \in (0, \infty)$, there exists $N := N(\varepsilon) \in \mathbb{Z}_+$, depending on ε , such that, for any $\ell, m \geq N$,

$$\|a^m - a^\ell\|_{\langle A_0, A_1 \rangle_\Theta} < \varepsilon/4.$$

Hence, there exists $\{b_i^{m, \ell}\}_{i \in \mathbb{Z}} \subset A_0 \cap A_1$ such that

$$a^m - a^\ell = \sum_{i=-\infty}^{\infty} b_i^{m, \ell} \quad \text{converges in } A_0 + A_1,$$

$\sum_{i=-\infty}^{\infty} \varepsilon_i 2^{i(j-\Theta)} b_i^{m, \ell}$ converges in A_j , $j \in \{0, 1\}$, and

$$\left\| \sum_{i=-\infty}^{\infty} \varepsilon_i 2^{i(j-\Theta)} b_i^{m, \ell} \right\|_{A_j} \leq \frac{\varepsilon}{2} \sup_{i \in \mathbb{Z}} |\varepsilon_i| \quad (4.5)$$

for any bounded sequence $\{\varepsilon_i\}_{i \in \mathbb{Z}}$. By choosing $\varepsilon_i := \delta_{i, k}$, $i \in \mathbb{Z}$, for fixed k , where $\delta_{i, k} := 1$ if $i = k$, otherwise $\delta_{i, k} := 0$, we know that, for any $i \in \mathbb{Z}$, $\|b_i^{m, \ell}\|_{A_j} < 2^{-i(j-\Theta)} \varepsilon/2$, $j \in \{0, 1\}$.

On the other hand, since $a^N \in \langle A_0, A_1 \rangle_\Theta$, it follows that there exists $\{a_i^N\}_{i \in \mathbb{Z}} \subset A_0 \cap A_1$ such that

$$a^N = \sum_{i=-\infty}^{\infty} a_i^N \quad \text{converges in } A_0 + A_1,$$

$\sum_{i=-\infty}^{\infty} \varepsilon_i 2^{i(j-\Theta)} a_i^N$ converges in A_j , $j \in \{0, 1\}$, and

$$\left\| \sum_{i=-\infty}^{\infty} \varepsilon_i 2^{i(j-\Theta)} a_i^N \right\|_{X_j} \leq (\|a^N\|_{\langle A_0, A_1 \rangle_\Theta} + \delta) \sup_{i \in \mathbb{Z}} |\varepsilon_i| \quad (4.6)$$

for any bounded sequence $\{\varepsilon_i\}_{i \in \mathbb{Z}}$, where $\delta \in (0, \infty)$ can be chosen as small as we want. Thus, for any $m \geq N$,

$$a^m = a^N + \sum_{i=-\infty}^{\infty} b_i^{m, N} = \sum_{i=-\infty}^{\infty} (a_i^N + b_i^{m, N}) =: \sum_{i=-\infty}^{\infty} \tilde{a}_i^m \quad \text{converge in } A_0 + A_1,$$

where $\{\tilde{a}_i^m\}_{i \in \mathbb{Z}} \subset A_0 \cap A_1$. Moreover, by (4.5) and (4.6), we see that $\sum_{i=-\infty}^{\infty} \varepsilon_i 2^{i(j-\Theta)} \tilde{a}_i^m$ converges in A_j , $j \in \{0, 1\}$, and

$$\left\| \sum_{i=-\infty}^{\infty} \varepsilon_i 2^{i(j-\Theta)} \tilde{a}_i^m \right\|_{A_j} \leq (\|a^N\|_{\langle A_0, A_1 \rangle_\Theta} + \delta + \varepsilon/2) \sup_{i \in \mathbb{Z}} |\varepsilon_i| \quad (4.7)$$

for any bounded sequence $\{\varepsilon_i\}_{i \in \mathbb{Z}}$. Since, for any $i \in \mathbb{Z}$ and $m, \ell \geq N$,

$$\|\tilde{a}_i^m - \tilde{a}_i^\ell\|_{X_j} \leq \|b_i^{m,N}\|_{A_j} + \|b_i^{\ell,N}\|_{A_j} < 2^{-i(j-\Theta)}\varepsilon, \quad j \in \{0, 1\},$$

we know that $\{\tilde{a}_i^m\}_m$ is a Cauchy sequence in $A_0 \cap A_1$. By the completeness of A_0 and A_1 , we obtain the existence of some $a_i := \lim_{m \rightarrow \infty} \tilde{a}_i^m$ in $A_0 \cap A_1$.

Since $a^m = \sum_{i=-\infty}^{\infty} \tilde{a}_i^m$ in $A_0 + A_1$, for any $\varepsilon \in (0, \infty)$, there exists $L \in \mathbb{Z}_+$ such that, for any $k > L$, $\|\sum_{L \leq |i| \leq k} \tilde{a}_i^m\|_{A_0+A_1} < \varepsilon$, which, together with the fact that $\tilde{a}_i^m \rightarrow a_i$ in $A_0 \cap A_1$, implies that $\|\sum_{L \leq |i| \leq k} a_i\|_{A_0+A_1} \leq \varepsilon$. Thus, $a := \sum_{i=-\infty}^{\infty} a_i$ converges in $A_0 + A_1$. Similarly, by (4.7), we find that $\sum_{i=-\infty}^{\infty} \varepsilon_i 2^{i(j-\Theta)} a_i$ converges in A_j , $j \in \{0, 1\}$, as well as

$$\left\| \sum_{i=-\infty}^{\infty} \varepsilon_i 2^{i(j-\Theta)} a_i \right\|_{A_j} \lesssim \sup_{i \in \mathbb{Z}} |\varepsilon_i|$$

for any bounded sequence $\{\varepsilon_i\}_{i \in \mathbb{Z}}$. Moreover, by the definition of a_i and (4.5), we know that

$$\|a - a^m\|_{\langle A_0, A_1 \rangle_\Theta} < \varepsilon$$

whenever $m \geq N$, namely, a^m converges to a in $\langle A_0, A_1 \rangle_\Theta$ as $m \rightarrow \infty$. This finishes the proof for the completeness of $\langle A_0, A_1 \rangle_\Theta$ in (i).

Next we show (ii). Notice that, if the summation $\sum_{i=-\infty}^{\infty} \varepsilon_i 2^{i(j-\Theta)} a_i$ converges in A_j , then

$$\sum_{i=-\infty}^{\infty} \varepsilon_i 2^{i(j-\Theta)} T a_i$$

converges in B_j , $j \in \{0, 1\}$, due to the boundedness of $T: A_j \rightarrow B_j$. Then, from

$$\begin{aligned} \left\| \sum_{i=-\infty}^{\infty} \varepsilon_i 2^{i(j-\Theta)} T a_i \right\|_{B_j} &= \left\| T \left(\sum_{i=-\infty}^{\infty} \varepsilon_i 2^{i(j-\Theta)} a_i \right) \right\|_{B_j} \\ &\leq \max\{\|T\|_{A_0 \rightarrow B_0}, \|T\|_{A_1 \rightarrow B_1}\} \left\| \sum_{i=-\infty}^{\infty} \varepsilon_i 2^{i(j-\Theta)} a_i \right\|_{A_j} \\ &\lesssim \max\{\|T\|_{A_0 \rightarrow B_0}, \|T\|_{A_1 \rightarrow B_1}\} \sup_{i \in \mathbb{Z}} |\varepsilon_i|, \end{aligned}$$

we deduce the desired conclusion in (ii), which completes the proof of Proposition 2.17.

Proof of Lemma 2.25

Step 1. Proof of (i). *Substep 1.1.* It is known that

$$\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n) = A_{p,q}^{s,\tau}(\mathbb{R}^n) \iff \tau = 0 \quad \text{and} \quad \max\{p, q\} < \infty,$$

$A \in \{B, F\}$; see [96, Theorem 2.3.3] for $\tau = 0$ and [111] for $\tau \in (0, \infty)$. This implies $\dot{A}_{p,q}^s(\mathbb{R}^n) = \dot{A}_{p,q}^s(\mathbb{R}^n)$ if $\max\{p, q\} < \infty$.

Substep 1.2. We prove $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n) \subsetneq \dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ if $\tau \in [1/p, \infty)$ and $p \in (0, \infty)$.

We study properties of the function $g(x) \equiv 1$ for all $x \in \mathbb{R}^n$. Let $\{\varphi_j\}_{j \in \mathbb{Z}_+}$ be the smooth decomposition of unity as defined in (5.1) and (5.2). By the basic properties of both the Fourier transform and our smooth decomposition of unity, we see that, for all $x \in \mathbb{R}^n$,

$$\mathcal{F}^{-1}(\varphi_j \mathcal{F}g)(x) = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that

$$\|g\|_{B_{p,q}^{s,\tau}(\mathbb{R}^n)} = \sup_{P \in \mathcal{Q}, |P| \geq 1} \frac{|P|^{1/p}}{|P|^\tau}$$

for any τ , p , q and s as in Lemma 2.25(i). Hence, $g \in B_{p,q}^{s,\tau}(\mathbb{R}^n)$ if and only if $\tau \in [1/p, \infty)$. Since all derivatives of g equal to zero, we further know that $g \in \dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Now let $f \in C_c^\infty(\mathbb{R}^n)$. We may assume $\text{supp } f \subset [-2^N, 2^N]^n$ for some $N \in \mathbb{N}$. Let ϕ denote the scaling function in Proposition 5.8. Now we choose a cube P such that $P = Q_{0,m}$ and

$$\text{dist}(Q_{0,m}, [-2^N, 2^N]^n) > N_2,$$

where N_2 is as in (5.3). Then Proposition 5.8 yields

$$\|g - f\|_{B_{p,q}^{s,\tau}(\mathbb{R}^n)} \geq |\langle g - f, \phi_{0,m} \rangle| = \left| \int_{\mathbb{R}^n} \phi(x) dx \right| > 0.$$

Hence, the function $g \equiv 1$ belongs to $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ but not to $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ if $\tau \in [1/p, \infty)$. It is easy to see that these statements remain true if we replace $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$, respectively, by $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Substep 1.3. Now we prove $\dot{A}_{p,\infty}^s(\mathbb{R}^n) = \dot{A}_{p,\infty}^s(\mathbb{R}^n)$ when $p \in (0, \infty)$. Obviously, we have $\dot{A}_{p,\infty}^s(\mathbb{R}^n) \hookrightarrow \dot{A}_{p,\infty}^s(\mathbb{R}^n)$.

To see the converse, let $\{\varphi_j\}_{j \in \mathbb{Z}_+}$ be the smooth decomposition of unity as defined in (5.1) and (5.2). By the Paley-Wiener theorem, for any $f \in \mathcal{S}'(\mathbb{R}^n)$ and any $j \in \mathbb{Z}_+$, the convolution $\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)$ is a smooth function, i.e., an infinitely differentiable function. Hence, also the function $S_N f$, defined by

$$S_N f(x) := \sum_{j=0}^N \mathcal{F}^{-1}(\varphi_j \mathcal{F}f)(x), \quad x \in \mathbb{R}^n, \quad N \in \mathbb{Z}_+, \quad (4.8)$$

is a smooth function. We claim that

(a) $\dot{B}_{p,\infty}^s(\mathbb{R}^n)$ is the collection of all $f \in B_{p,\infty}^s(\mathbb{R}^n)$ such that

$$\lim_{j \rightarrow \infty} 2^{js} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)\|_{L_p(\mathbb{R}^n)} = 0;$$

(b) $\dot{F}_{p,\infty}^s(\mathbb{R}^n)$ is the collection of all $f \in F_{p,\infty}^s(\mathbb{R}^n)$ such that

$$\lim_{N \rightarrow \infty} \left\| \sup_{j \geq N} 2^{js} |\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)| \right\|_{L_p(\mathbb{R}^n)} = 0.$$

To prove these claims, we argue as follows. For simplicity, we concentrate on the F -case. Using some standard Fourier multiplier assertions (see [96, 2.3.7]), embeddings (see [96, 2.7.1]) and the lifting properties (see [96, 2.3.8]), it is easily seen that $f \in F_{p,\infty}^s(\mathbb{R}^n)$ implies that $S_N f \in F_{p,q}^\sigma(\mathbb{R}^n)$ for all $N \in \mathbb{Z}_+$, all $\sigma \in \mathbb{R}$ and all $q \in (0, \infty]$. For $f \in \dot{F}_{p,\infty}^s(\mathbb{R}^n)$, let $\{f_\ell\}_{\ell \in \mathbb{N}}$ be a sequence such that $D^\alpha f_\ell \in F_{p,\infty}^s(\mathbb{R}^n)$ for all $\alpha \in (\mathbb{Z}_+)^n$ and $\ell \in \mathbb{N}$, and

$$\lim_{\ell \rightarrow \infty} \|f - f_\ell\|_{F_{p,\infty}^s(\mathbb{R}^n)} = 0.$$

Then, for any $\varepsilon \in (0, \infty)$, there exists $N \in \mathbb{N}$, depending on ε , such that $\|f - f_\ell\|_{F_{p,\infty}^s(\mathbb{R}^n)} < \varepsilon$ whenever $\ell \geq N$. Observe that $f_\ell \in F_{p,q}^\sigma(\mathbb{R}^n)$ for all $\sigma \in \mathbb{R}$, all q and all ℓ (see [96, 2.3.8]). On the other hand, it holds true that

$$\begin{aligned} \|f - S_N f_\ell\|_{F_{p,\infty}^s(\mathbb{R}^n)} &= \left\| \sum_{j=N}^{\infty} \mathcal{F}^{-1}(\varphi_j \mathcal{F}f_\ell) \right\|_{F_{p,\infty}^s(\mathbb{R}^n)} \lesssim \left\| \sup_{j \geq N} 2^{js} |\mathcal{F}^{-1}(\varphi_j \mathcal{F}f_\ell)| \right\|_{L_p(\mathbb{R}^n)} \\ &\lesssim 2^{-N(\sigma-s)} \left\| \sup_{j \geq N} 2^{j\sigma} |\mathcal{F}^{-1}(\varphi_j \mathcal{F}f_\ell)| \right\|_{L_p(\mathbb{R}^n)} \\ &\lesssim 2^{-N(\sigma-s)} \|f_\ell\|_{F_{p,\infty}^s(\mathbb{R}^n)} \end{aligned}$$

for all $\sigma > s$. The implicit positive constants in these inequalities are independent of ℓ . Therefore, by this and the boundedness in $F_{p,\infty}^s(\mathbb{R}^n)$ of S_N uniformly in $N \in \mathbb{N}$, with $\kappa := \min\{1, p\}$, we have

$$\begin{aligned} \|f - S_N f\|_{F_{p,\infty}^s(\mathbb{R}^n)}^\kappa &\leq \|f - f_\ell\|_{F_{p,\infty}^s(\mathbb{R}^n)}^\kappa + \|f_\ell - S_N f_\ell\|_{F_{p,\infty}^s(\mathbb{R}^n)}^\kappa + \|S_N f_\ell - S_N f\|_{F_{p,\infty}^s(\mathbb{R}^n)}^\kappa \\ &\lesssim \|f - f_\ell\|_{F_{p,\infty}^s(\mathbb{R}^n)}^\kappa + \|f_\ell - S_N f_\ell\|_{F_{p,\infty}^s(\mathbb{R}^n)}^\kappa \\ &\lesssim \|f - f_\ell\|_{F_{p,\infty}^s(\mathbb{R}^n)}^\kappa + 2^{-N\kappa(\sigma-s)} \|f_\ell\|_{F_{p,\infty}^s(\mathbb{R}^n)}^\kappa, \end{aligned}$$

which tends to 0 as $N \rightarrow \infty$, since $\{\|f_\ell\|_{F_{p,\infty}^s(\mathbb{R}^n)}\}_\ell$ is bounded. Hence $f = \lim_{N \rightarrow \infty} S_N f$ in $F_{p,\infty}^s(\mathbb{R}^n)$. The claim (b) then follows from the observation that

$$\|f - S_N f\|_{F_{p,\infty}^s(\mathbb{R}^n)} \lesssim \left\| \sup_{j \geq N} 2^{js} |\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)| \right\|_{L_p(\mathbb{R}^n)} \lesssim \|f - S_{N-1} f\|_{F_{p,\infty}^s(\mathbb{R}^n)}$$

with the implicit positive constants independent of N .

The proof of claim (a) is similar, the details being omitted.

Now we are ready to prove $\dot{A}_{p,\infty}^s(\mathbb{R}^n) = \dot{A}_{p,\infty}^s(\mathbb{R}^n)$. Let $f \in \dot{A}_{p,\infty}^s(\mathbb{R}^n)$. By the above claims, it suffices to approximate $S_N f$ by functions from $C_c^\infty(\mathbb{R}^n)$. Indeed, let ψ be as in (5.1). Then

$$\lim_{M \rightarrow \infty} \psi(2^{-M} x) S_N f(x) = S_N f(x)$$

with convergence in $A_{p,\infty}^s(\mathbb{R}^n)$, as desired.

Substep 1.4. We now show that $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n) \subsetneq \dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ when $\tau \in (0, 1/p)$. We use functions defined by a wavelet series (see Appendix and, particularly, Subsection 5.4). Let

$$f := \sum_{\ell=1}^{\infty} \phi_{0,(2^\ell, 0, \dots, 0)},$$

where ϕ is the scaling function in Proposition 5.8. Now we show $f \in A_{p,q}^{s,\tau}(\mathbb{R}^n)$ by using Proposition 5.8. First we see that

$$\sum_{i=1}^{2^n-1} \|\{\langle f, \psi_{i,j,k} \rangle\}_{j,k}\|_{a_{p,q}^{s,\tau}(\mathbb{R}^n)} = 0.$$

It remains to estimate

$$\sup_{P \in \mathcal{Q}, |P| \geq 1} \frac{1}{|P|^\tau} \left(\sum_{Q_{0,m} \subset P} |\langle f, \phi_{0,m} \rangle|^p \right)^{1/p}.$$

An inspection of the supports of the functions $\phi_{0,(2^\ell, 0, \dots, 0)}$ makes clear that it suffices to consider either cubes $P \in \mathcal{Q}$ with volume 1 or cubes $P \in \mathcal{Q}$ with $P = P_M := [0, 2^M]^n$, $M \in \mathbb{N}$. Concentrating on the second case, we obtain

$$\sup_{M \in \mathbb{N}} \frac{1}{2^{\tau M n}} \left(\sum_{0 \leq m_j < 2^M, j \in \{1, \dots, n\}} |\langle f, \phi_{0,m} \rangle|^p \right)^{1/p} \lesssim \sup_{M \in \mathbb{N}} \frac{1}{2^{\tau M n}} M^{1/p} \lesssim 1,$$

because $\tau \in (0, \infty)$. If $P \in \mathcal{Q}$ with volume 1, then $P = Q_{0,m}$ for some $m \in \mathbb{Z}^n$ and, in this case,

$$\frac{1}{|P|^\tau} \left(\sum_{Q_{0,m} \subset P} |\langle f, \phi_{0,m} \rangle|^p \right)^{1/p} \asymp 1.$$

Thus, $f \in A_{p,q}^{s,\tau}(\mathbb{R}^n)$. Since the support of f is not compact, it is easily shown by using Proposition 5.8 that $f \notin \dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$; see also Substep 1.2 of this proof.

Now we construct an approximation of f by smooth functions. The function ϕ can be approximated by its Sobolev mollification, denoted by $\phi^{(\varepsilon)}$, in the norm of $C^{N_1-1}(\mathbb{R}^n)$. To explain the notation, let ω

be an infinitely differentiable function such that $\text{supp } \omega \subset B(0, 1)$, $\omega \geq 0$ and $\int_{\mathbb{R}^n} \omega(x) dx = 1$. Then, for $\varepsilon \in (0, \infty)$, we put

$$\phi^{(\varepsilon)}(x) := \varepsilon^{-n} \int_{\mathbb{R}^n} \omega\left(\frac{x-y}{\varepsilon}\right) \phi(y) dy, \quad x \in \mathbb{R}^n, \quad (4.9)$$

as well as

$$f_\varepsilon := \sum_{\ell=1}^{\infty} \phi_{0, (2^\ell, 0, \dots, 0)}^{(\varepsilon)}.$$

Clearly, $f_\varepsilon \in C^\infty(\mathbb{R}^n)$. For all $\alpha \in (\mathbb{Z}_+)^n$ with $|\alpha| < N_1$, it follows that, for all $x \in \mathbb{R}^n$,

$$|D^\alpha f(x) - D^\alpha f_\varepsilon(x)| \leq \varepsilon^{-n} \int_{\mathbb{R}^n} \omega\left(\frac{x-y}{\varepsilon}\right) |D^\alpha \phi(y) - D^\alpha \phi(x)| dy \leq \sqrt{n} \|\phi\|_{C^{N_1}(\mathbb{R}^n)} \varepsilon, \quad (4.10)$$

where

$$\|\phi\|_{C^N(\mathbb{R}^n)} := \max_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} |D^\alpha \phi(x)|.$$

We claim $f_\varepsilon \in B_{p,q}^{s,\tau}(\mathbb{R}^n)$ and

$$\lim_{\varepsilon \downarrow 0} \|f - f_\varepsilon\|_{B_{p,q}^{s,\tau}(\mathbb{R}^n)} = 0. \quad (4.11)$$

Due to the definitions of f and f_ε , and the lifting property (see [115, Proposition 5.1]), it suffices to prove this claim for s large and $q = \infty$. In such a situation, we may use the characterization of $B_{p,q}^{s,\tau}(\mathbb{R}^n)$ by differences as proved in [115, 4.3.2].

Let $p \in [1, \infty]$, $q \in (0, \infty]$ and

$$0 < s \leq \max\{s, s + n\tau - n/p\} < M$$

with $M \in \mathbb{N}$. We define

$$\|f\|_{B_{p,\infty}^{s,\tau}(\mathbb{R}^n)}^\clubsuit := \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \sup_{0 < t < 2 \min\{\ell(P), 1\}} t^{-s} \left(\int_P [a_t(f, x)]^p dx \right)^{1/p},$$

where $\ell(P)$ denotes the side-length of the cube P ,

$$a_t(f, x) := t^{-n} \int_{t/2 \leq |h| < t} |\Delta_h^M f(x)| dh, \quad x \in \mathbb{R}^n,$$

and $\Delta_h^M f$ denotes the M -th difference of f . In addition, we put

$$\|f\|_{L_p^\tau(\mathbb{R}^n)} := \sup_{P \in \mathcal{Q}, |P| \geq 1} \frac{1}{|P|^\tau} \left[\int_P |f(x)|^p dx \right]^{1/p}.$$

Then, from [115, 4.3.2], we deduce that $f \in B_{p,\infty}^{s,\tau}(\mathbb{R}^n)$ if and only if $f \in L_p^\tau(\mathbb{R}^n)$ and $\|f\|_{B_{p,\infty}^{s,\tau}(\mathbb{R}^n)}^\clubsuit < \infty$.

Furthermore, $\|f\|_{L_p^\tau(\mathbb{R}^n)} + \|f\|_{B_{p,\infty}^{s,\tau}(\mathbb{R}^n)}^\clubsuit$ and $\|f\|_{B_{p,\infty}^{s,\tau}(\mathbb{R}^n)}$ are equivalent.

As a consequence of (4.10), we find that

$$\|f - f_\varepsilon\|_{L_p^\tau(\mathbb{R}^n)} \leq \sqrt{n} \|\phi\|_{C^{N_1}(\mathbb{R}^n)} \varepsilon.$$

Now we investigate $\|f - f_\varepsilon\|_{B_{p,\infty}^{s,\tau}(\mathbb{R}^n)}^\clubsuit$. Let $M < N_1$, where N_1 is as in (5.3). Since, for all $x \in \mathbb{R}^n$,

$$t^{-n} \int_{t/2 \leq |h| < t} |\Delta_h^M (f - f_\varepsilon)(x)| dh \lesssim \varepsilon t^M,$$

we conclude, for small cubes P (i.e., $|P| \leq 1$), that

$$\sup_{P \in \mathcal{Q}, |P| \leq 1} \frac{1}{|P|^\tau} \sup_{0 < t < 2 \min\{\ell(P), 1\}} t^{-s} \left(\int_P [\varepsilon t^M]^p dx \right)^{1/p} \lesssim \varepsilon \sup_{P \in \mathcal{Q}, |P| \leq 1} \frac{|P|^{1/p} [\ell(P)]^{M-s}}{|P|^\tau} \lesssim \varepsilon.$$

Within the large cubes P , it suffices to consider $P_L = [0, 2^L]^n$, $L \in \mathbb{N}$. Then, by the support condition of f and f_ε , we see that

$$\sup_{L \in \mathbb{N}} \frac{1}{2^{Ln\tau}} \sup_{0 < t < 2} t^{-s} \left(\sum_{\ell=1}^L \int_{Q_{0,(2^\ell, 0, \dots, 0)}} [\varepsilon t^M]^p dx \right)^{1/p} \lesssim \varepsilon \sup_{L \in \mathbb{N}} \frac{L^{1/p}}{2^{Ln\tau}} \lesssim \varepsilon. \quad (4.12)$$

Combining these two inequalities, we show the above claim (4.11) in case $p \in [1, \infty]$.

For $p \in (0, 1)$, we can argue in principal as above. However, a few modifications are necessary, since, in this case, the characterization by differences looks a bit different. Let $p \in (0, 1)$, $q \in (0, \infty]$ and

$$n \left(\frac{1}{p} - 1 \right) < s \leq \max\{s, s + n\tau - n/p\} < M$$

with $M \in \mathbb{N}$. Let s_0 be chosen such that $n(1/p - 1) < s_0 < s$. Then $f \in B_{p,\infty}^{s,\tau}(\mathbb{R}^n)$ if and only if $f \in L_\tau^p(\mathbb{R}^n)$, $\|f\|_{B_{p,\infty}^{s,\tau}(\mathbb{R}^n)}^\clubsuit < \infty$ and

$$\sup_{P \in \mathcal{Q}, |P| \geq 1} \frac{\|f\|_{B_{p,\infty}^{s_0}(2P)}}{|P|^\tau} < \infty, \quad (4.13)$$

where $B_{p,\infty}^{s_0}(2P)$ is defined as in Definition 5.12 below. Furthermore,

$$\|f\|_{L_\tau^p(\mathbb{R}^n)} + \|f\|_{B_{p,\infty}^{s,\tau}(\mathbb{R}^n)}^\clubsuit + \sup_{P \in \mathcal{Q}, |P| \geq 1} \frac{\|f\|_{B_{p,\infty}^{s_0}(2P)}}{|P|^\tau} \quad \text{and} \quad \|f\|_{B_{p,\infty}^{s,\tau}(\mathbb{R}^n)}$$

are equivalent. The additional term in (4.13) can be treated as in (4.12), the details being omitted. Thus, (4.11) also holds true in this case. By this and $f \in B_{p,q}^{s,\tau}(\mathbb{R}^n)$, we further conclude that $f \in \dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$. Recall that it is proved, in Substep 1.4, that $f \notin \dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$. Thus, we have $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n) \subsetneq \dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ when $\tau \in (0, 1/p)$.

The F -case can be derived from $B_{p,\min(p,q)}^{s,\tau}(\mathbb{R}^n) \hookrightarrow F_{p,q}^{s,\tau}(\mathbb{R}^n) \hookrightarrow B_{p,\max(p,q)}^{s,\tau}(\mathbb{R}^n)$, the details being omitted.

Step 2. Proof of (ii). In case $u = p$, we have $\mathcal{N}_{p,p,q}^s(\mathbb{R}^n) = B_{p,q}^s(\mathbb{R}^n)$. With $p \in (0, \infty)$, Step 1 implies that

$$\dot{\mathcal{N}}_{p,p,q}^s(\mathbb{R}^n) = \dot{B}_{p,q}^s(\mathbb{R}^n) = \dot{B}_{p,q}^s(\mathbb{R}^n) = \dot{\mathcal{N}}_{p,p,q}^s(\mathbb{R}^n).$$

The non-coincidence when $0 < p < u < \infty$ can be proved by an argument parallel to that used in Step 1 for the spaces $A_{p,q}^{s,\tau}(\mathbb{R}^n)$. We refer the reader to Proposition 5.11 for the wavelet characterization of $\mathcal{N}_{p,p,q}^s(\mathbb{R}^n)$ and to [115, 4.5.2] for an appropriate characterization of $\mathcal{N}_{p,p,q}^s(\mathbb{R}^n)$ by differences, the details being omitted. This finishes the proof of Lemma 2.25.

Proof of Lemma 2.26

Step 1. Proof of (i). We prove this by considering four cases.

Substep 1.1. $\tau \in (1/p, \infty)$. In this case,

$$A_{p,q}^{s,\tau}(\mathbb{R}^n) = B_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n), \quad A \in \{B, F\};$$

see Proposition 5.2(iii) below. Hence, in this case, to show Lemma 2.26(i), it suffices to prove that $\dot{B}_{\infty,\infty}^s(\mathbb{R}^n)$ is a proper subspace of $B_{\infty,\infty}^s(\mathbb{R}^n)$. Temporarily we assume $s \in (0, 1)$. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $\psi(x) = 1$ if $|x| \leq 1$. Then we define

$$f_\alpha(x) := \psi(x) |x|^\alpha, \quad x \in \mathbb{R}^n.$$

Clearly, $f_\alpha \in B_{\infty,\infty}^s(\mathbb{R}^n)$ if and only if $\alpha \geq s$; see, e.g., [73, Lemma 2.3.1/1]. We claim that $f_s \in B_{\infty,\infty}^s(\mathbb{R}^n) \setminus \dot{B}_{\infty,\infty}^s(\mathbb{R}^n)$. When $s \in (0, 1)$, since $B_{\infty,\infty}^s(\mathbb{R}^n)$ is just the Lipschitz space of order s (see, for example, [96, 2.3.5]), it follows that

$$\|f\|_{B_{\infty,\infty}^s(\mathbb{R}^n)} \geq \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^s}.$$

Notice that, for all smooth functions g satisfying that $D^\alpha g \in B_{\infty,\infty}^s(\mathbb{R}^n)$ for all $\alpha \in \mathbb{Z}_+^n$, by the lifting property of $B_{\infty,\infty}^s(\mathbb{R}^n)$ (see [96, Theorem 2.3.8]), we conclude that $g \in B_{\infty,\infty}^\sigma(\mathbb{R}^n)$ for all $\sigma \in (s, \infty)$ and hence

$$\lim_{y \rightarrow 0} \frac{|g(0) - g(y)|}{|y|^s} \leq \lim_{y \rightarrow 0} |y|^{\sigma-s} \|g\|_{B_{\infty,\infty}^\sigma(\mathbb{R}^n)} = 0.$$

Since this kind of functions is dense in $\dot{B}_{\infty,\infty}^s(\mathbb{R}^n)$, the above assertion holds true also for all $g \in \dot{B}_{\infty,\infty}^s(\mathbb{R}^n)$. On the other hand,

$$\frac{|f_s(0) - f_s(y)|}{|y|^s} = 1 \quad \text{for all } y \neq 0, \quad |y| \leq 1.$$

This implies $\|f_s - g\|_{B_{\infty,\infty}^s(\mathbb{R}^n)} \geq 1$ for all $g \in \dot{B}_{\infty,\infty}^s(\mathbb{R}^n)$, and proves the above claim.

Now we remove the restriction $s \in (0, 1)$ by assuming $s \in \mathbb{R}$. In this general case, we apply the lifting operator

$$I_\sigma : f \mapsto \mathcal{F}^{-1} \left((1 + |\cdot|^2)^{\sigma/2} \mathcal{F}f(\cdot) \right), \quad f \in \mathcal{S}'(\mathbb{R}^n).$$

Here σ is a real number. It is well known that I_σ is an isomorphism which maps $B_{\infty,\infty}^s(\mathbb{R}^n)$ onto $B_{\infty,\infty}^{s-\sigma}(\mathbb{R}^n)$; see [96, 2.3.8]. In addition, I_σ maps $\dot{B}_{\infty,\infty}^s(\mathbb{R}^n)$ onto $\dot{B}_{\infty,\infty}^{s-\sigma}(\mathbb{R}^n)$. By using this, we can transfer the problem to the case $s \in (0, 1)$. This shows that the above claim also holds true for all $s \in \mathbb{R}$. *Substep 1.2.* $\tau = 1/p > 0$. In this case, we have to prove that $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is a proper subspace of $A_{p,q}^{s,\tau}(\mathbb{R}^n)$. This time we use wavelet representations of $A_{p,q}^{s,\tau}(\mathbb{R}^n)$ (see Proposition 5.8 in Appendix). Let

$$f_s := \sum_{j=1}^{\infty} 2^{-j(s+n/2)} \psi_{1,j,(0,\dots,0)}.$$

Proposition 5.8 yields

$$\|f_s\|_{B_{p,q}^{s,\tau}(\mathbb{R}^n)} \asymp \sup_{P \subset Q_{0,0}} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} 2^{j(s+\frac{n}{2})q} \left[\int_P 2^{-j(s+n/2)p} \chi_{Q_{j,0}}(x) dx \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}}.$$

Observe that $\sup_{P \subset Q_{0,0}} \cdots = \sup_{\ell \in \mathbb{Z}_+, P=Q_{\ell,0}} \cdots$. This implies, in case $p < \infty$, that

$$\|f_s\|_{B_{p,q}^{s,\tau}(\mathbb{R}^n)} \asymp \sup_{\ell \in \mathbb{Z}_+} 2^{\ell n/p} \left\{ \sum_{j=\ell}^{\infty} 2^{-jnq/p} \right\}^{\frac{1}{q}} \lesssim 1.$$

Hence, $f_s \in A_{p,q}^{s,\tau}(\mathbb{R}^n)$ for all $q \in (0, \infty]$ and all $p \in (0, \infty)$, $A \in \{B, F\}$ (the F -case follows from $B_{p,\min(p,q)}^{s,\tau}(\mathbb{R}^n) \hookrightarrow F_{p,q}^{s,\tau}(\mathbb{R}^n) \hookrightarrow B_{p,\max(p,q)}^{s,\tau}(\mathbb{R}^n)$).

We claim that $f_s \notin \dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$. The proof makes use of the continuous embedding

$$A_{p,q}^{s,\tau}(\mathbb{R}^n) \hookrightarrow B_{\infty,\infty}^{s+n\tau-n/p}(\mathbb{R}^n);$$

see [115, Proposition 2.6]. From this embedding, we immediately derive that

$$\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n) \hookrightarrow \dot{B}_{\infty,\infty}^{s+n\tau-n/p}(\mathbb{R}^n) = \dot{B}_{\infty,\infty}^s(\mathbb{R}^n),$$

since $\tau = 1/p$.

Recall that the wavelet characterization of $\dot{B}_{\infty,\infty}^s(\mathbb{R}^n)$ looks as follows (see Remark 5.9): Let $s \in \mathbb{R}$ and $N_1 > s$. Then $f \in \dot{B}_{\infty,\infty}^s(\mathbb{R}^n)$ if and only if f can be represented as in (5.4) (with convergence in $\mathcal{S}'(\mathbb{R}^n)$), $\|\Phi(f)\|_{b_{\infty,\infty}^s(\mathbb{R}^n)}^* < \infty$ and

$$\lim_{j \rightarrow \infty} 2^{j(s+n/2)} \max_{i \in \{1, \dots, 2^n - 1\}} \sup_{k \in \mathbb{Z}^n} |\langle f, \psi_{i,j,k} \rangle| = 0.$$

From the definition of f_s , it follows obviously that $f_s \notin \dot{B}_{\infty,\infty}^s(\mathbb{R}^n)$. Consequently, $f_s \notin \dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and hence $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n) \subsetneq A_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Substep 1.3. $\tau \in (0, 1/p)$ and $q \in (0, \infty]$. In this case, we argue as in Substep 1.2. Let

$$f_{s,p,\tau} := \sum_{j=1}^{\infty} 2^{-j(s+n(\tau-1/p)+n/2)} \psi_{1,j,(0,\dots,0)}. \quad (4.14)$$

Proposition 5.8 yields

$$\|f_{s,p,\tau}\|_{B_{p,q}^{s,\tau}(\mathbb{R}^n)} \asymp \sup_{P \subset Q_{0,0}} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} 2^{j(s+\frac{n}{p})q} \left[\int_P 2^{-j(s+n(\tau-1/p)+n/2)p} \chi_{Q_{j,0}}(x) dx \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}}.$$

As above, this implies, in case $\tau \in (0, \infty)$, that

$$\|f_{s,p,\tau}\|_{B_{p,q}^{s,\tau}(\mathbb{R}^n)} \asymp \sup_{\ell \in \mathbb{Z}_+} 2^{\ell n \tau} \left\{ \sum_{j=\ell}^{\infty} 2^{-jn\tau q} \right\}^{\frac{1}{q}} \lesssim 1.$$

Hence, $f_{s,p,\tau} \in A_{p,q}^{s,\tau}(\mathbb{R}^n)$ for all $q \in (0, \infty]$, $A \in \{B, F\}$ (again the F -case follows from $B_{p,\min(p,q)}^{s,\tau}(\mathbb{R}^n) \hookrightarrow F_{p,q}^{s,\tau}(\mathbb{R}^n) \hookrightarrow B_{p,\max(p,q)}^{s,\tau}(\mathbb{R}^n)$). Since $f_{s,p,\tau} \notin \dot{B}_{\infty,\infty}^{s+n\tau-n/p}(\mathbb{R}^n)$ (see Substep 1.2.), we conclude $f_{s,p,\tau} \notin \dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and hence $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n) \subsetneq A_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Substep 1.4. $\tau = 0$. In this case, it is known that

$$\dot{A}_{p,q}^s(\mathbb{R}^n) = A_{p,q}^s(\mathbb{R}^n) \iff \max\{p, q\} < \infty;$$

see [96, Theorem 2.3.3]. Hence, by Lemma 2.25, we know that $\dot{A}_{p,q}^s(\mathbb{R}^n) = A_{p,q}^s(\mathbb{R}^n)$ if $\max\{p, q\} < \infty$.

Let $p < \infty$ and $q = \infty$. Again, by Lemma 2.25, we have $\dot{A}_{p,\infty}^s(\mathbb{R}^n) = \dot{A}_{p,\infty}^s(\mathbb{R}^n)$, and the latter is known to be a proper subspace of $A_{p,\infty}^s(\mathbb{R}^n)$.

Let $p = \infty$ and $q < \infty$. Then $\dot{B}_{\infty,q}^s(\mathbb{R}^n) = B_{\infty,q}^s(\mathbb{R}^n)$, since

$$\lim_{N \rightarrow \infty} \|f - S_N f\|_{B_{\infty,q}^s(\mathbb{R}^n)} \asymp \lim_{N \rightarrow \infty} \left\{ \sum_{j=N}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)\|_{L_{\infty}(\mathbb{R}^n)}^q \right\}^{1/q} = 0.$$

Finally, $\dot{B}_{\infty,\infty}^s(\mathbb{R}^n)$ is a proper subset of $B_{\infty,\infty}^s(\mathbb{R}^n)$, since f_s in Substep 1.2 belongs to $B_{\infty,\infty}^s(\mathbb{R}^n)$ and it does not belong to $\dot{B}_{\infty,\infty}^s(\mathbb{R}^n)$.

This finishes the proof of (i).

Step 2. Proof of (ii). To show this, we consider two cases.

Substep 2.1. Let $0 < p \leq u < \infty$ and $q \in (0, \infty)$. Let $S_N f$ be defined as in (4.8). Using the Fourier multiplier theorem of Tang and Xu [92], it is not difficult to prove

$$\|f - S_N f\|_{\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)} \lesssim \left(\sum_{j=N}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)\|_{\mathcal{M}_p^u(\mathbb{R}^n)}^q \right)^{1/q} \rightarrow 0$$

if N tends to infinity. This implies $\dot{\mathcal{N}}_{u,p,q}^s(\mathbb{R}^n) = \mathcal{N}_{u,p,q}^s(\mathbb{R}^n)$ for $q \in (0, \infty)$.

Substep 2.2. Let $0 < p \leq u < \infty$ and $q = \infty$. Because of $\mathcal{N}_{u,p,\infty}^s(\mathbb{R}^n) = B_{p,\infty}^{s, \frac{1}{p} - \frac{1}{u}}(\mathbb{R}^n)$, the desired conclusion follows from Step 1. This finishes the proof of Lemma 2.26.

Remark 4.9. It is of certain interest to point out that Proposition 5.11 implies that the function $f_{s,p,\tau}$, defined in (4.14), does not belong to any of the spaces $\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)$, $1/u := 1/p - \tau$, $\tau \in (0, 1/p)$ and $q \in (0, \infty)$. This implies that

$$B_{p,q_0}^{s, \frac{1}{p} - \frac{1}{u}}(\mathbb{R}^n) \hookrightarrow \mathcal{N}_{u,p,q_1}^s(\mathbb{R}^n) \iff q_1 = \infty$$

and

$$F_{p,q_0}^{s, \frac{1}{p} - \frac{1}{q}}(\mathbb{R}^n) \hookrightarrow \mathcal{N}_{u,p,q_1}^s(\mathbb{R}^n) \implies q_1 = \infty,$$

which have been known before, and we refer the reader to Sawano [75].

Proof of Lemma 2.27

Step 1. Proof of (i). We need the following modification of Proposition 2.7 (see [111]), here and hereafter, $\dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n)$ denotes the closure of finite sequences in $a_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Proposition 4.10. *Let $\Theta \in (0, 1)$, $s, s_0, s_1 \in \mathbb{R}$, $\tau, \tau_0, \tau_1 \in [0, \infty)$, $p, p_0, p_1 \in (0, \infty]$ and $q, q_0, q_1 \in (0, \infty]$ such that $s = s_0(1 - \Theta) + s_1\Theta$, $\tau = \tau_0(1 - \Theta) + \tau_1\Theta$, $\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$ and $\frac{1}{q} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}$. If $\tau_0 p_0 = \tau_1 p_1$, then*

$$[\dot{a}_{p_0,q_0}^{s_0,\tau_0}(\mathbb{R}^n)]^{1-\Theta} [\dot{a}_{p_1,q_1}^{s_1,\tau_1}(\mathbb{R}^n)]^\Theta = \dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n), \quad a \in \{f, b\}.$$

Thanks to Proposition 4.3, we obtain

$$\langle \dot{a}_{p_0,q_0}^{s_0,\tau_0}(\mathbb{R}^n), \dot{a}_{p_1,q_1}^{s_1,\tau_1}(\mathbb{R}^n) \rangle_\Theta = \overline{\dot{a}_{p_0,q_0}^{s_0,\tau_0}(\mathbb{R}^n) \cap \dot{a}_{p_1,q_1}^{s_1,\tau_1}(\mathbb{R}^n)}^{\|\cdot\|_{a_{p,q}^{s,\tau}(\mathbb{R}^n)}}.$$

Since all finite sequences belong to $\dot{a}_{p_0,q_0}^{s_0,\tau_0}(\mathbb{R}^n) \cap \dot{a}_{p_1,q_1}^{s_1,\tau_1}(\mathbb{R}^n)$, it is clear that the closure must be $\dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n)$. By means of Propositions 5.8 and 2.17, this carries over to the function spaces. Consequently, it holds true that

$$\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n) = \left\langle \dot{A}_{p_0,q_0}^{s_0,\tau_0}(\mathbb{R}^n), \dot{A}_{p_1,q_1}^{s_1,\tau_1}(\mathbb{R}^n) \right\rangle_\Theta \hookrightarrow \langle A_{p_0,q_0}^{s_0,\tau_0}(\mathbb{R}^n), A_{p_1,q_1}^{s_1,\tau_1}(\mathbb{R}^n) \rangle_\Theta.$$

Concerning the remaining embedding, we recall

$$A_{p,q}^{s,\tau}(\mathbb{R}^n) \hookrightarrow B_{\infty,\infty}^{s+n\tau-n/p}(\mathbb{R}^n)$$

(see [115, Proposition 2.6]), and therefore

$$\langle A_{p_0,q_0}^{s_0,\tau_0}(\mathbb{R}^n), A_{p_1,q_1}^{s_1,\tau_1}(\mathbb{R}^n) \rangle_\Theta \hookrightarrow \langle B_{\infty,\infty}^{s_0+n\tau_0-n/p_0}(\mathbb{R}^n), B_{\infty,\infty}^{s_1+n\tau_1-n/p_1}(\mathbb{R}^n) \rangle_\Theta.$$

To calculate $\langle B_{\infty,\infty}^{s_0+n\tau_0-n/p_0}(\mathbb{R}^n), B_{\infty,\infty}^{s_1+n\tau_1-n/p_1}(\mathbb{R}^n) \rangle_\Theta$, we use the identity

$$\begin{aligned} & \left\langle B_{\infty,\infty}^{s_0+n\tau_0-n/p_0}(\mathbb{R}^n), B_{\infty,\infty}^{s_1+n\tau_1-n/p_1}(\mathbb{R}^n) \right\rangle_\Theta \\ &= \left(B_{\infty,\infty}^{s_0+n\tau_0-n/p_0}(\mathbb{R}^n), B_{\infty,\infty}^{s_1+n\tau_1-n/p_1}(\mathbb{R}^n), B_{\infty,\infty}^{s+n\tau-n/p}(\mathbb{R}^n), \# \right) \end{aligned}$$

in Proposition 2.21(i). Then, it suffices to show

$$(B_{\infty,\infty}^{s_0}(\mathbb{R}^n), B_{\infty,\infty}^{s_1}(\mathbb{R}^n), B_{\infty,\infty}^s(\mathbb{R}^n), \#) = \dot{B}_{\infty,\infty}^s(\mathbb{R}^n), \quad (4.15)$$

if $s_0 \neq s_1$, $s := s_0(1 - \Theta) + s_1\Theta$. To see this, without loss of generality, we may assume $s_0 < s_1$. Then $B_{\infty,\infty}^{s_1}(\mathbb{R}^n) \hookrightarrow B_{\infty,\infty}^{s_0}(\mathbb{R}^n)$ and

$$B_{\infty,\infty}^{s_1}(\mathbb{R}^n) \cap B_{\infty,\infty}^{s_0}(\mathbb{R}^n) = B_{\infty,\infty}^{s_1}(\mathbb{R}^n)$$

follows. Applying a simple lifting argument (see Theorem 2.3.8 in [96]), we find that

$$\begin{aligned} & \{f \in B_{\infty,\infty}^{s_1}(\mathbb{R}^n) : D^\alpha f \in B_{\infty,\infty}^{s_1}(\mathbb{R}^n) \text{ for all } \alpha \in \mathbb{Z}_+\} \\ &= \{f \in B_{\infty,\infty}^s(\mathbb{R}^n) : D^\alpha f \in B_{\infty,\infty}^s(\mathbb{R}^n) \text{ for all } \alpha \in \mathbb{Z}_+\}. \end{aligned}$$

Hence

$$\dot{B}_{\infty,\infty}^s(\mathbb{R}^n) \hookrightarrow \overline{B_{\infty,\infty}^{s_1}(\mathbb{R}^n)}^{\|\cdot\|_{B_{\infty,\infty}^s(\mathbb{R}^n)}}.$$

To prove the converse, let $f \in B_{\infty,\infty}^{s_1}(\mathbb{R}^n)$. Then

$$\|f - S_N f\|_{B_{\infty,\infty}^s(\mathbb{R}^n)} \lesssim \sup_{j \geq N} 2^{js} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)\|_{L_\infty(\mathbb{R}^n)}$$

$$\begin{aligned} &\lesssim 2^{-N(s_1-s)} \sup_{j \geq N} 2^{js_1} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)\|_{L^\infty(\mathbb{R}^n)} \\ &\lesssim 2^{-N(s_1-s)} \|f\|_{B_{\infty,\infty}^{s_1}(\mathbb{R}^n)} \rightarrow 0 \end{aligned}$$

as $N \rightarrow \infty$. This proves $f \in \dot{B}_{\infty,\infty}^s(\mathbb{R}^n)$ and therefore, the claim (4.15) is established.

Step 2. Proof of (ii). This time we need the following modification of Proposition 2.8 (see [111]).

Proposition 4.11. *Let $\Theta \in (0, 1)$, $s, s_0, s_1 \in \mathbb{R}$, $q, q_0, q_1 \in (0, \infty]$, $0 < p \leq u \leq \infty$, $0 < p_0 \leq u_0 \leq \infty$ and $0 < p_1 \leq u_1 \leq \infty$ such that $\frac{1}{q} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}$, $\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}$, $s = s_0(1-\Theta) + s_1\Theta$ and $\frac{1}{u} = \frac{1-\Theta}{u_0} + \frac{\Theta}{u_1}$. If $p_0 u_1 = p_1 u_0$, then*

$$[\hat{n}_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n)]^{1-\Theta} [\hat{n}_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n)]^\Theta = \hat{n}_{u, p, q}^s(\mathbb{R}^n).$$

Now we can proceed as in Step 1 since, with $\tau := \frac{1}{p} - \frac{1}{u}$,

$$\mathcal{N}_{u, p, q}^s(\mathbb{R}^n) \hookrightarrow \mathcal{N}_{u, p, \infty}^s(\mathbb{R}^n) = B_{p, \infty}^{s, \tau}(\mathbb{R}^n) \hookrightarrow B_{\infty, \infty}^{s+n\tau-n/p}(\mathbb{R}^n).$$

Step 3. Proof of (iii). Clearly, by (2.6), we know that $\langle A_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), A_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n) \rangle_\Theta$ is a subspace of $A_{p, q}^{s, \tau}(\mathbb{R}^n)$. If $\langle A_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), A_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n) \rangle_\Theta$ would coincide with $A_{p, q}^{s, \tau}(\mathbb{R}^n)$, Step 1 would imply that

$$A_{p, q}^{s, \tau}(\mathbb{R}^n) \hookrightarrow \dot{B}_{\infty, \infty}^{s+n\tau-n/p}(\mathbb{R}^n).$$

But, in (4.14), we have found a function $f_{s, p, \tau}$ such that $f_{s, p, \tau} \in A_{p, q}^{s, \tau}(\mathbb{R}^n) \setminus \dot{B}_{\infty, \infty}^{s+n\tau-n/p}(\mathbb{R}^n)$ if $\tau \in (0, 1/p)$. Thus,

$$\langle A_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), A_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n) \rangle_\Theta \subsetneq A_{p, q}^{s, \tau}(\mathbb{R}^n),$$

which completes the proof of Lemma 2.27.

Proof of Theorem 2.28

Step 1. Proof of (i). Without loss of generality, we may assume $p_0 + q_0 < \infty$. By Propositions 2.7 and 4.3, we see that

$$\langle a_{p_0, q_0}^{s_0}(\mathbb{R}^n), a_{p_1, q_1}^{s_1}(\mathbb{R}^n) \rangle_\Theta = \overline{a_{p_0, q_0}^{s_0}(\mathbb{R}^n) \cap a_{p_1, q_1}^{s_1}(\mathbb{R}^n)}^{\|\cdot\|_{a_{p, q}^s(\mathbb{R}^n)}}.$$

Since finite sequences are contained in $a_{p_0, q_0}^{s_0}(\mathbb{R}^n) \cap a_{p_1, q_1}^{s_1}(\mathbb{R}^n)$, and dense in $a_{p, q}^s(\mathbb{R}^n)$ due to $p + q < \infty$, it follows that

$$\overline{a_{p_0, q_0}^{s_0}(\mathbb{R}^n) \cap a_{p_1, q_1}^{s_1}(\mathbb{R}^n)}^{\|\cdot\|_{a_{p, q}^s(\mathbb{R}^n)}} = a_{p, q}^s(\mathbb{R}^n).$$

Based on Propositions 2.17 and 5.8, this equality can be transferred to the related function spaces. The equivalence with $\hat{A}_{p, q}^s(\mathbb{R}^n)$ and $\dot{A}_{p, q}^s(\mathbb{R}^n)$ follows from Lemmas 2.25 and 2.26.

Step 2. Proof of (ii). Let $s_0 \neq s_1$. Then the case $q_0 = q_1 = q = \infty$ has been treated above (see (4.15)). The general case $q_0, q_1 \in (0, \infty]$ can be handled in the same way.

Now we turn to the case $s = s_0 = s_1$ and $q_0 < q < q_1$. Observe that

$$B_{\infty, q_0}^s(\mathbb{R}^n) \cap B_{\infty, q_1}^s(\mathbb{R}^n) = B_{\infty, q_0}^s(\mathbb{R}^n).$$

Hence, we need to calculate the closure of $B_{\infty, q_0}^s(\mathbb{R}^n)$ in $B_{\infty, q}^s(\mathbb{R}^n)$. Clearly, for $f \in B_{\infty, q}^s(\mathbb{R}^n)$, we have $S_N f \in B_{\infty, q_0}^s(\mathbb{R}^n)$ and

$$\begin{aligned} \|f - S_N f\|_{B_{\infty, q}^s(\mathbb{R}^n)} &\lesssim \left[\sum_{j=N}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)\|_{L^\infty(\mathbb{R}^n)}^q \right]^{1/q} \\ &\lesssim \left[\sum_{j=N}^{\infty} 2^{jsq_0} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)\|_{L^\infty(\mathbb{R}^n)}^{q_0} \right]^{1/q_0}. \end{aligned}$$

Since the right-hand side of the above inequalities tends to 0 as $N \rightarrow \infty$, we conclude that

$$\langle B_{\infty, q_0}^s(\mathbb{R}^n), B_{\infty, q_1}^s(\mathbb{R}^n) \rangle_\Theta = \overline{B_{\infty, q_0}^s(\mathbb{R}^n)}^{\|\cdot\|_{B_{\infty, q}^s(\mathbb{R}^n)}} = B_{\infty, q}^s(\mathbb{R}^n).$$

Step 3. Proof of (iii). We follow the proof of the claim in (4.15). Without loss of generality, we may assume $s_0 < s_1$. Then $A_{p,\infty}^{s_1}(\mathbb{R}^n) \hookrightarrow A_{p,\infty}^{s_0}(\mathbb{R}^n)$ and

$$A_{p,\infty}^{s_1}(\mathbb{R}^n) \cap A_{p,\infty}^{s_0}(\mathbb{R}^n) = A_{p,\infty}^{s_1}(\mathbb{R}^n)$$

follows. Again a simple lifting argument (see Theorem 2.3.8 in [96]) yields

$$\begin{aligned} \{f \in A_{p,\infty}^{s_1}(\mathbb{R}^n) : D^\alpha f \in A_{p,\infty}^{s_1}(\mathbb{R}^n) \text{ for all } \alpha \in \mathbb{Z}_+\} \\ = \{A_{p,\infty}^s(\mathbb{R}^n) : D^\alpha f \in A_{p,\infty}^s(\mathbb{R}^n) \text{ for all } \alpha \in \mathbb{Z}_+\}. \end{aligned}$$

Hence

$$\mathring{A}_{p,\infty}^s(\mathbb{R}^n) \hookrightarrow \overline{A_{p,\infty}^{s_1}(\mathbb{R}^n)}^{\|\cdot\|_{A_{p,\infty}^s(\mathbb{R}^n)}}.$$

To prove the converse, let $f \in A_{p,\infty}^{s_1}(\mathbb{R}^n)$. Then

$$\|f - S_N f\|_{A_{p,\infty}^s(\mathbb{R}^n)} \lesssim 2^{-N(s_1-s)} \|f\|_{A_{p,\infty}^{s_1}(\mathbb{R}^n)}.$$

This implies $f \in \mathring{A}_{p,\infty}^s(\mathbb{R}^n)$, and hence

$$\overline{A_{p,\infty}^{s_0}(\mathbb{R}^n) \cap A_{p,\infty}^{s_1}(\mathbb{R}^n)}^{\|\cdot\|_{A_{p,\infty}^s(\mathbb{R}^n)}} = \mathring{A}_{p,\infty}^s(\mathbb{R}^n).$$

Step 4. Proofs of (iv) and (v). We follow some arguments taken over from [87] where a similar situation for the complex method has been treated. By similarity, we concentrate on the F -case. Notice that our assumptions are guaranteeing

$$F_{p_0,\infty}^{s_0}(\mathbb{R}^n) \hookrightarrow F_{p,1}^s(\mathbb{R}^n) \hookrightarrow F_{p_1,\infty}^{s_1}(\mathbb{R}^n);$$

see [96, Theorem 2.7.1]. This implies that

$$F_{p_0,\infty}^{s_0}(\mathbb{R}^n) \cap F_{p_1,\infty}^{s_1}(\mathbb{R}^n) = F_{p_0,\infty}^{s_0}(\mathbb{R}^n).$$

Clearly,

$$\overline{C_c^\infty(\mathbb{R}^n)}^{\|\cdot\|_{F_{p,\infty}^s(\mathbb{R}^n)}} \hookrightarrow \overline{F_{p_0,\infty}^{s_0}(\mathbb{R}^n)}^{\|\cdot\|_{F_{p,\infty}^s(\mathbb{R}^n)}} \hookrightarrow \overline{F_{p,1}^s(\mathbb{R}^n)}^{\|\cdot\|_{F_{p,\infty}^s(\mathbb{R}^n)}}$$

Since $C_c^\infty(\mathbb{R}^n)$ is dense in $F_{p,1}^s(\mathbb{R}^n)$, we know that all three spaces coincide with $\mathring{F}_{p,\infty}^s(\mathbb{R}^n)$. But, in Lemma 2.25, we prove, in this situation, $\mathring{F}_{p,\infty}^s(\mathbb{R}^n) = \mathring{F}_{p,\infty}^s(\mathbb{R}^n)$.

Step 5. Proof of (vi). Again, we follow [87], where a similar situation for the complex method is treated. It suffices to argue on the level of sequence spaces due to Proposition 5.8. For $j \in \mathbb{Z}_+$, let K_j be a subset of \mathbb{Z}^n with cardinality

$$|K_j| = \lceil 2^{-j\{(s_1-s_0) \cdot \frac{1}{1/p_1-1/p_0} - d\}} \rceil,$$

where $\lceil t \rceil$ denotes the smallest integer larger than or equal to $t \in \mathbb{R}$. We define a sequence $\lambda := \{\lambda_{j,k}\}_{j,k}$ by

$$\lambda_{j,k} := \begin{cases} 2^{j \cdot \frac{p_1 s_1 - p_0 s_0}{p_0 - p_1}} & \text{if } k \in K_j, j \in \mathbb{Z}_+, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily checked that

$$\lambda \in b_{p,\infty}^{s_0,0}(\mathbb{R}^n) \setminus \mathring{b}_{p,\infty}^{s_0,0}(\mathbb{R}^n) \quad \text{and} \quad \lambda \in b_{p_0,\infty}^{s_0,0}(\mathbb{R}^n) \cap b_{p_1,\infty}^{s_1,0}(\mathbb{R}^n).$$

The counterpart of Proposition 2.21 on the sequence space level yields

$$\lambda \in \overline{b_{p_0,\infty}^{s_0,0}(\mathbb{R}^n) \cap b_{p_1,\infty}^{s_1,0}(\mathbb{R}^n)}^{b_{p,\infty}^s(\mathbb{R}^n)} = \langle b_{p_0,\infty}^{s_0,0}(\mathbb{R}^n), b_{p_1,\infty}^{s_1,0}(\mathbb{R}^n) \rangle_\Theta.$$

Hence, the embedding $\mathring{b}_{p,\infty}^{s_0,0}(\mathbb{R}^n) \hookrightarrow \langle b_{p_0,\infty}^{s_0,0}(\mathbb{R}^n), b_{p_1,\infty}^{s_1,0}(\mathbb{R}^n) \rangle_\Theta$, as well as its function space version, is strict in this case. Moreover, $\langle B_{p_0,\infty}^{s_0}(\mathbb{R}^n), B_{p_1,\infty}^{s_1}(\mathbb{R}^n) \rangle_\Theta \subsetneq B_{p,\infty}^s(\mathbb{R}^n)$ is a consequence of Lemma 2.27(i).

Step 6. Proof of (vii). We proceed as in Step 4. Observe that this time $\max\{p, q\} < \infty$. Thanks to the embedding $B_{p_0, \infty}^{s_0}(\mathbb{R}^n) \hookrightarrow B_{\infty, q_1}^{s_1}(\mathbb{R}^n)$ (see [96, 2.7.1]), we know that

$$C_c^\infty(\mathbb{R}^n) \hookrightarrow B_{p_0, \infty}^{s_0}(\mathbb{R}^n) \cap B_{\infty, q_1}^{s_1}(\mathbb{R}^n) = B_{p_0, \infty}^{s_0}(\mathbb{R}^n).$$

On the other hand, using $B_{p_0, \infty}^{s_0}(\mathbb{R}^n) \hookrightarrow B_{p, q}^s(\mathbb{R}^n)$ (see [96, 2.7.1]), we find that

$$\overline{C_c^\infty(\mathbb{R}^n)}^{B_{p, q}^{s_1}(\mathbb{R}^n)} \hookrightarrow \overline{B_{p_0, \infty}^{s_0}(\mathbb{R}^n)}^{B_{p, q}^{s_1}(\mathbb{R}^n)} \hookrightarrow \overline{B_{p, q}^s(\mathbb{R}^n)}^{B_{p, q}^{s_1}(\mathbb{R}^n)} = B_{p, q}^s(\mathbb{R}^n).$$

Lemma 2.25(i) shows that the space on the left-hand side of the above formula coincides with $B_{p, q}^s(\mathbb{R}^n)$. The proof of Theorem 2.28 is then complete.

Proof of Theorem 2.29

By Proposition 2.20, we have to calculate

$$\overline{A_{p, q_0}^{s_0, \tau}(\mathbb{R}^n) \cap A_{p, q_1}^{s_1, \tau}(\mathbb{R}^n)}^{\|\cdot\|_{A_{p, q}^{s, \tau}(\mathbb{R}^n)}}.$$

Step 1. First, we assume $A = B$ and $s_0 > s_1$. Clearly, $B_{p, q_0}^{s_0, \tau}(\mathbb{R}^n) \cap B_{p, q_1}^{s_1, \tau}(\mathbb{R}^n) = B_{p, q_0}^{s_0, \tau}(\mathbb{R}^n)$. We claim

$$\overline{B_{p, q_0}^{s_0, \tau}(\mathbb{R}^n)}^{\|\cdot\|_{B_{p, q}^{s, \tau}(\mathbb{R}^n)}} = \dot{B}_{p, q}^{s, \tau}(\mathbb{R}^n).$$

But this follows immediately from

$$\begin{aligned} & \left\{ f \in B_{p, q_0}^{s_0, \tau}(\mathbb{R}^n) : D^\alpha f \in B_{p, q_0}^{s_0, \tau}(\mathbb{R}^n) \text{ for all } \alpha \in \mathbb{Z}_+^n \right\} \\ &= \left\{ f \in B_{p, q}^{s, \tau}(\mathbb{R}^n) : D^\alpha f \in B_{p, q}^{s, \tau}(\mathbb{R}^n) \text{ for all } \alpha \in \mathbb{Z}_+^n \right\}. \end{aligned} \quad (4.16)$$

To prove this identity, we need the following result: Let $m \in \mathbb{N}$. Then $A_{p, q}^{s, \tau}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\sum_{|\alpha|=m} \|D^\alpha f\|_{A_{p, q}^{s-m, \tau}(\mathbb{R}^n)} < \infty$$

in the sense of equivalent quasi-norms. If $A = F$, by Propositions 5.2 and 5.5, this property of $F_{p, q}^{s, \tau}(\mathbb{R}^n)$ when $\tau = 0$ or $\tau \in [1/p, \infty)$ can be found in [96, 2.3.8], while when $\tau \in (0, 1/p)$ was proved by Tang and Xu [92]; If $A = B$, by Proposition 5.2 again, this property of $B_{p, q}^{s, \tau}(\mathbb{R}^n)$ when $\tau = 0$ or $\tau \in (1/p, \infty)$ was proved in [96, 2.3.8], while when $\tau \in (0, 1/p]$ can be proved by an argument similar to that used in the proof of [92, Theorem 2.15(ii)].

In addition, we mention the embedding

$$A_{p, q}^{s, \tau}(\mathbb{R}^n) \hookrightarrow C_{ub}(\mathbb{R}^n) \quad \text{if } s + n\tau - n/p > 0$$

(see [115, Proposition 2.6]), where $C_{ub}(\mathbb{R}^n)$ denotes the space of uniformly continuous and bounded functions on \mathbb{R}^n . This, together with the above property of $A_{p, q}^{s, \tau}(\mathbb{R}^n)$, shows that both sets in (4.16) contain only $C^\infty(\mathbb{R}^n)$ functions and hence they are equal. Thus, the above claim $\overline{B_{p, q_0}^{s_0, \tau}(\mathbb{R}^n)}^{\|\cdot\|_{B_{p, q}^{s, \tau}(\mathbb{R}^n)}} = \dot{B}_{p, q}^{s, \tau}(\mathbb{R}^n)$ holds true. This proves the desired conclusion of Theorem 2.29 in this case.

Step 2. This time we consider the case $A = B$, $s = s_0 = s_1$ and $q_0 < q < q_1$. Clearly, this time

$$B_{p, q_0}^{s, \tau}(\mathbb{R}^n) \cap B_{p, q_1}^{s, \tau}(\mathbb{R}^n) = B_{p, q_0}^{s, \tau}(\mathbb{R}^n)$$

and we can argue as in Step 1.

Step 3. Also, if $A = F$, we can argue as in Step 1, the details being omitted. This finishes the proof of Theorem 2.29.

Proof of Theorem 2.30

Under the assumptions of Theorem 2.30, we see that

$$\begin{aligned} \langle A_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), \mathcal{A}_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n) \rangle_{\Theta} &= \left\langle B_{\infty, \infty}^{s_0+n(\tau_0-1/p_0)}(\mathbb{R}^n), B_{\infty, \infty}^{s_1+n(\tau_1-1/p_1)}(\mathbb{R}^n) \right\rangle_{\Theta} \\ &= \mathring{B}_{\infty, \infty}^{s_0+n(\tau_0-1/p)}(\mathbb{R}^n) = \mathring{A}_{p, q}^{s, \tau}(\mathbb{R}^n); \end{aligned}$$

see Proposition 5.2 in Appendix and (4.15). This finishes the proof of Theorem 2.30.

Proof of Theorem 2.32

Clearly, it follows from Proposition 2.21 that

$$\langle \mathcal{N}_{u, p, q_0}^{s_0}(\mathbb{R}^n), \mathcal{N}_{u, p, q_1}^{s_1}(\mathbb{R}^n) \rangle_{\Theta} = \overline{\mathcal{N}_{u, p, q_0}^{s_0}(\mathbb{R}^n) \cap \mathcal{N}_{u, p, q_1}^{s_1}(\mathbb{R}^n)}^{\|\cdot\|_{\mathcal{N}_{u, p, q}^s(\mathbb{R}^n)}}.$$

For fixed u and p , we put

$$N_q^s(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) : D^\alpha f \in \mathcal{N}_{u, p, q}^s(\mathbb{R}^n) \text{ for all } \alpha \in \mathbb{Z}_+^n \right\}.$$

If $s_0 > s_1$, then the obvious embedding $\mathcal{N}_{u, p, q_0}^{s_0}(\mathbb{R}^n) \hookrightarrow \mathcal{N}_{u, p, q_1}^{s_1}(\mathbb{R}^n)$ implies that

$$N_{q_0}^{s_0}(\mathbb{R}^n) = N_{q_1}^{s_1}(\mathbb{R}^n) = N_q^s(\mathbb{R}^n).$$

However, also in case $s_0 = s_1$, we have the coincidence of these spaces independent of q_0, q_1 , due to an argument similar to that used in the proof for (4.16). On the other hand, Lemma 2.26 yields $\mathring{\mathcal{N}}_{u, p, q_i}^{s_i}(\mathbb{R}^n) = \mathcal{N}_{u, p, q_i}^{s_i}(\mathbb{R}^n)$, $i \in \{0, 1\}$. All above observations, together with Proposition 2.21, further implies that

$$\begin{aligned} \langle \mathcal{N}_{u, p, q_0}^{s_0}(\mathbb{R}^n), \mathcal{N}_{u, p, q_1}^{s_1}(\mathbb{R}^n) \rangle_{\Theta} &\hookleftarrow \overline{N_{q_0}^{s_0}(\mathbb{R}^n) \cap N_{q_1}^{s_1}(\mathbb{R}^n)}^{\|\cdot\|_{\mathcal{N}_{u, p, q}^s(\mathbb{R}^n)}} = \overline{N_q^s(\mathbb{R}^n)}^{\|\cdot\|_{\mathcal{N}_{u, p, q}^s(\mathbb{R}^n)}} \\ &= \mathring{\mathcal{N}}_{u, p, q}^s(\mathbb{R}^n) = \mathcal{N}_{u, p, q}^s(\mathbb{R}^n) \hookrightarrow \langle \mathcal{N}_{u, p, q_0}^{s_0}(\mathbb{R}^n), \mathcal{N}_{u, p, q_1}^{s_1}(\mathbb{R}^n) \rangle_{\Theta}, \end{aligned}$$

which implies the desired conclusion of Theorem 2.32 and hence completes its proof.

Proof of Lemma 2.33

Step 1. Proof of (i). Let $M(\mathbb{R}^n)$ denote the space of all functions $g \in \mathcal{M}_p^u(\mathbb{R}^n)$ having the properties (2.10), (2.11) (uniformly in $y \in \mathbb{R}^n$) and (2.12) (uniformly in $r \in (0, \infty)$).

We first show $\mathring{\mathcal{M}}_p^u(\mathbb{R}^n) \hookrightarrow M(\mathbb{R}^n)$. Obviously, all smooth compactly supported functions satisfy the conditions (2.10) through (2.12). Let now $g \in \mathring{\mathcal{M}}_p^u(\mathbb{R}^n)$ and $\{f_\ell\}_\ell \in C_c^\infty(\mathbb{R}^n)$ be an approximating sequence of g in $\mathcal{M}_p^u(\mathbb{R}^n)$. Then

$$\begin{aligned} |B(y, r)|^{1/u-1/p} \left[\int_{B(y, r)} |g(x)|^p dx \right]^{1/p} &\leq |B(y, r)|^{1/u-1/p} \left[\int_{B(y, r)} |g(x) - f_\ell(x)|^p dx \right]^{1/p} \\ &\quad + |B(y, r)|^{1/u-1/p} \left[\int_{B(y, r)} |f_\ell(x)|^p dx \right]^{1/p}. \end{aligned} \quad (4.17)$$

For any given $\varepsilon \in (0, 1)$, since $f_\ell \rightarrow g$ in $\mathcal{M}_p^u(\mathbb{R}^n)$ as $\ell \rightarrow \infty$, we choose ℓ so large such that

$$|B(y, r)|^{1/u-1/p} \left[\int_{B(y, r)} |g(x) - f_\ell(x)|^p dx \right]^{1/p} \leq \frac{\varepsilon}{2}$$

(simultaneously for all r and all y). Next, by f_ℓ satisfying (2.11), we choose r , depending on the already chosen ℓ but independent of y , so large such that

$$|B(y, r)|^{1/u-1/p} \left[\int_{B(y, r)} |f_\ell(x)|^p dx \right]^{1/p} \leq \frac{\varepsilon}{2}.$$

Consequently,

$$|B(y, r)|^{1/u-1/p} \left[\int_{B(y, r)} |g(x)|^p dx \right]^{1/p} \leq \varepsilon,$$

if r is large enough and independent of y . This proves that g has the property (2.11).

Now we turn to show that g satisfies (2.10). Again we make use of (4.17). Then, instead of choosing r large, we have to choose r small enough. From (4.17) and $\{f_\ell\}_{\ell \in \mathbb{N}}$ satisfying (2.10), together with an argument similar to the above, it follows that

$$|B(y, r)|^{1/u-1/p} \left[\int_{B(y, r)} |g(x)|^p dx \right]^{1/p} \leq \varepsilon,$$

if r is small enough and independent of y . This means, g also satisfies (2.10).

It remains to prove that g satisfies (2.12). By (4.17) again, together with $\{f_\ell\}_{\ell \in \mathbb{N}}$ satisfying (2.12) and an argument similar to above, we see that

$$|B(y, r)|^{1/u-1/p} \left[\int_{B(y, r)} |g(x)|^p dx \right]^{1/p} \leq \varepsilon,$$

if $|y| \geq N := N(\varepsilon)$ is large enough. Thus, $g \in M(\mathbb{R}^n)$. Altogether this proves that $\mathcal{M}_p^u(\mathbb{R}^n) \hookrightarrow M(\mathbb{R}^n)$.

To show the converse, we suppose $g \in M(\mathbb{R}^n)$. For such g , we have to prove the existence of a sequence of smooth compactly supported functions approximating g in the norm of $\mathcal{M}_p^u(\mathbb{R}^n)$. We prove this by three steps.

Substep 1.1. For $g \in M(\mathbb{R}^n)$, we wish to approximate g by uniformly continuous functions. In this first step, we will find it convenient to replace the balls in the definition of $\|\cdot\|_{\mathcal{M}_p^u(\mathbb{R}^n)}$ by dyadic cubes. This results in an equivalent norm, and the convergence is not influenced.

Let

$$N(t) := \begin{cases} t & \text{if } 0 \leq t \leq 1, \\ 2-t & \text{if } 1 \leq t \leq 2, \\ 0 & \text{otherwise} \end{cases}$$

be the standard hat function (B -spline of order 2). Its tensor product is denoted by

$$\overline{N}(x) := \prod_{i=1}^n N(x_i) \quad \text{for all } x := (x_1, \dots, x_n).$$

Then the integer shifts of \overline{N} form a decomposition of unity, i.e.,

$$\sum_{k \in \mathbb{Z}^n} \overline{N}(x - k) = 1 \quad \text{for all } x \in \mathbb{R}^n.$$

For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, let

$$Q_{j,k}^* := \{x \in \mathbb{R}^n : 2^{-j}k_i \leq x_i < 2^{-j}(k_i + 2), i \in \{1, \dots, n\}\}.$$

Observe, $\overline{Q_{j,k}^*} = \text{supp } \overline{N}(2^j \cdot -k)$. For $f \in L_1^{\text{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we put

$$T_j f(x) := \sum_{k \in \mathbb{Z}^n} 2^{n(j-1)} \left[\int_{Q_{j,k}^*} f(y) dy \right] \overline{N}(2^j x - k), \quad j \in \mathbb{Z}_+.$$

Many times, we use the following $L_p(\mathbb{R}^n)$ -stability of the integer translates of \overline{N} . For any dyadic cube $Q \subset \mathbb{R}^n$, it holds true that

$$\int_Q |T_j f(x)|^p dx = \int_Q \left| \sum_{k: Q_{j,k}^* \cap Q \neq \emptyset} 2^{n(j-1)} \left[\int_{Q_{j,k}^*} f(y) dy \right] \overline{N}(2^j x - k) \right|^p dx$$

$$\asymp \sum_{k: Q_{j,k}^* \cap Q \neq \emptyset} \int_{Q_{j,k}^*} \left| 2^{n(j-1)} \int_{Q_{j,k}^*} f(y) dy \right|^p dx \asymp \sum_{k: Q_{j,k}^* \cap Q \neq \emptyset} 2^{jn(p-1)} \left| \int_{Q_{j,k}^*} f(y) dy \right|^p.$$

Next we employ the Hölder inequality and find that

$$\left[\int_Q |T_j f(x)|^p dx \right]^{1/p} \lesssim \left[\sum_{k: Q_{j,k}^* \cap Q \neq \emptyset} 2^{jn(p-1)} 2^{-jnp/p'} \int_{Q_{j,k}^*} |f(y)|^p dy \right]^{1/p} \lesssim \left[\int_Q |f(y)|^p dy \right]^{1/p}.$$

Here the implicit positive constants are independent of f , Q and j . Replacing the dyadic cube by a cube of type, $\tilde{Q} := \Pi_{i=1}^n [y_i - 2^N, y_i + 2^N]$, $y \in \mathbb{R}^n$, we find that a similar estimate is true:

$$\left[\int_{\tilde{Q}} |T_j f(x)|^p dx \right]^{1/p} \lesssim \left[\int_{2\tilde{Q}} |f(y)|^p dy \right]^{1/p}, \quad (4.18)$$

where $2\tilde{Q}$ denotes the cube with the same center as \tilde{Q} , sides parallel to the sides of \tilde{Q} and side-length twice larger. The inequality (4.18) further implies that, if $g \in M(\mathbb{R}^n)$, also $T_j g$ belongs to $M(\mathbb{R}^n)$.

Now we prove that $T_j g$ converges to g in $\mathcal{M}_p^u(\mathbb{R}^n)$. We consider two cases:

(a) Let Q be a dyadic cube such that $|Q| \geq 2^{-jn}$. For brevity, we put

$$a_{j,k} := 2^{n(j-1)} \int_{Q_{j,k}^*} g(y) dy.$$

This implies that

$$\begin{aligned} \int_Q |g(x) - T_j g(x)|^p dx &= \int_Q \left| \sum_{k \in \mathbb{Z}^n: Q_{j,k}^* \cap Q \neq \emptyset} [g(x) - a_{j,k}] \overline{N}(2^j x - k) \right|^p dx \\ &\lesssim \sum_{k \in \mathbb{Z}^n: Q_{j,k}^* \cap Q \neq \emptyset} \int_{Q_{j,k}^*} |g(x) - a_{j,k}|^p dx. \end{aligned}$$

By the Hölder inequality, we see that

$$\begin{aligned} &\sum_{k \in \mathbb{Z}^n: Q_{j,k}^* \cap Q \neq \emptyset} \int_{Q_{j,k}^*} |g(x) - a_{j,k}|^p dx \\ &= \sum_{k \in \mathbb{Z}^n: Q_{j,k}^* \cap Q \neq \emptyset} \int_{Q_{j,k}^*} 2^{n(j-1)p} \left| \int_{Q_{j,k}^*} [g(x) - g(y)] dy \right|^p dx \\ &\leq 2^{jn} \sum_{k \in \mathbb{Z}^n: Q_{j,k}^* \cap Q \neq \emptyset} \int_{Q_{j,k}^*} \int_{Q_{j,k}^*} |g(x) - g(y)|^p dy dx \\ &= 2^{jn} \int_{Q_{j,0}^*} \sum_{k \in \mathbb{Z}^n: Q_{j,k}^* \cap Q \neq \emptyset} \int_{Q_{j,k}^*} |g(x + 2^{-j}k) - g(y)|^p dy dx \\ &\lesssim \sup_{|h| \leq \sqrt{n}2^{-j+1}} \sum_{k \in \mathbb{Z}^n: Q_{j,k}^* \cap Q \neq \emptyset} \int_{Q_{j,k}^*} |g(y+h) - g(y)|^p dy \\ &\lesssim \sup_{|h| \leq \sqrt{n}2^{-j+1}} \int_Q |g(y+h) - g(y)|^p dy. \end{aligned}$$

As above, we switch again to cubes of type, $\tilde{Q} := \Pi_{i=1}^n [y_i - 2^N, y_i + 2^N]$, $y \in \mathbb{R}^n$, and it follows that

$$\left[\int_{\tilde{Q}} |g(x) - T_j g(x)|^p dx \right]^{1/p} \lesssim \sup_{|h| \leq \sqrt{n}2^{-j+1}} \left[\int_{2\tilde{Q}} |g(y+h) - g(y)|^p dy \right]^{1/p}. \quad (4.19)$$

(b) Let \tilde{Q} be a cube of type, $\Pi_{i=1}^n [y_i - 2^N, y_i + 2^N]$, $y \in \mathbb{R}^n$, such that $|\tilde{Q}| < 2^{-jn}$. By obvious modifications of the above argument, we find, also in this case, that (4.19) holds true.

Now we use (4.19) to prove $\|g - T_j g\|_{\mathcal{M}_p^u(\mathbb{R}^n)} \rightarrow 0$ as $j \rightarrow \infty$. Let $\tau_h g(x) := g(x + h)$, $x \in \mathbb{R}^n$. Continuity of translations is well known in the context of $L_p(\mathbb{R}^n)$ -spaces. This will be also applied here. However, we need one more preparation. Since $g \in M(\mathbb{R}^n)$, it follows that, for any $\varepsilon \in (0, \infty)$, there exists $N_0 \in \mathbb{N}$, depending on ε , such that

$$2^{Nn(\frac{1}{u} - \frac{1}{p})} \left[\int_{\Pi_{i=1}^n [y_i - 2^N, y_i + 2^N]} |g(x)|^p dx \right]^{1/p} < \varepsilon \quad \text{for all } N \geq N_0, \quad (4.20)$$

uniformly in y . Similarly, there exists $N_1 \in \mathbb{N}$, depending on ε , such that

$$2^{-Nn(\frac{1}{u} - \frac{1}{p})} \left[\int_{\Pi_{i=1}^n [y_i - 2^{-N}, y_i + 2^{-N}]} |g(x)|^p dx \right]^{1/p} < \varepsilon \quad \text{for all } N \geq N_1, \quad (4.21)$$

uniformly in y . Using (4.18), it suffices to deal with those cubes \tilde{Q} such that $2^{-N_1 n} \leq |\tilde{Q}| \leq 2^{N_0 n}$. Then (4.19) leads to

$$\begin{aligned} & \sup_{-N_1 \leq N \leq N_0} 2^{Nn(\frac{1}{u} - \frac{1}{p})} \left[\int_{\Pi_{i=1}^n [y_i - 2^N, y_i + 2^N]} |g(x) - T_j g(x)|^p dx \right]^{1/p} \\ & \lesssim 2^{-N_1 n(\frac{1}{u} - \frac{1}{p})} \left[\int_{\Pi_{i=1}^n [y_i - 2^{N_0}, y_i + 2^{N_0}]} |g(x) - T_j g(x)|^p dx \right]^{1/p} \\ & \lesssim 2^{-N_1 n(\frac{1}{u} - \frac{1}{p})} \sup_{|h| \leq \sqrt{n} 2^{-j+1}} \left[\int_{\Pi_{i=1}^n [y_i - 2^{N_0}, y_i + 2^{N_0}]} |g(x) - g(x + h)|^p dx \right]^{1/p}. \end{aligned} \quad (4.22)$$

Since

$$\lim_{|h| \rightarrow 0} \|f(\cdot) - f(\cdot + h)\|_{L_p(\mathbb{R}^n)} = 0 \quad \text{for all } f \in L_p(\mathbb{R}^n),$$

the right-hand side in (4.22) tends to zero for j tending to ∞ . By the translation invariance of the $L_p(\mathbb{R}^n)$ -norm, this estimate is independent of y . Combining (4.20), (4.21) and (4.22), we see that, for all $g \in M(\mathbb{R}^n)$,

$$\|g - T_j g\|_{\mathcal{M}_p^u(\mathbb{R}^n)} \lesssim \varepsilon \quad (4.23)$$

if j is large enough. Observe that, for each $j \in \mathbb{Z}_+$, $T_j g$ is an uniformly continuous function on \mathbb{R}^n , since

$$\begin{aligned} 2^{n(j-1)} \left| \int_{Q_{j,k}^*} g(y) dy \right| & \lesssim 2^{jn/p} \left[\int_{Q_{j,k}^*} |g(y)|^p dy \right]^{1/p} \\ & \lesssim_j |Q_{j,k}^*|^{\frac{1}{u} - \frac{1}{p}} \left[\int_{Q_{j,k}^*} |g(y)|^p dy \right]^{1/p} \lesssim_j \|g\|_{\mathcal{M}_p^u(\mathbb{R}^n)}, \end{aligned}$$

where \lesssim_j denotes the implicit positive constants depending on j . Thus, this shows that $g \in M(\mathbb{R}^n)$ can be approximated by uniformly continuous functions in $\mathcal{M}_p^u(\mathbb{R}^n)$.

Substep 1.2. For $g \in M(\mathbb{R}^n)$, we wish to approximate $T_j g$ by C^∞ -functions. To this end, we use the Sobolev mollification (see (4.9)). To simplify notation, we denote $T_j g$ just by u . Consequently, for any given $\tilde{\varepsilon} \in (0, \infty)$, there exists $\delta \in (0, \infty)$, depending on $\tilde{\varepsilon}$, such that, if $|x - y| < \delta$, then $|u(x) - u(y)| < \tilde{\varepsilon}$. Thus, for given $\varepsilon \in (0, \infty)$, we see that

$$\sup_{-N_1 \leq N \leq N_0} 2^{Nn(\frac{1}{u} - \frac{1}{p})} \left[\int_{\Pi_{i=1}^n [y_i - 2^N, y_i + 2^N]} |u(x) - u^{(\delta)}(x)|^p dx \right]^{1/p}$$

$$\begin{aligned} &\lesssim 2^{-N_1 n(\frac{1}{u}-\frac{1}{p})} \left[\int_{\prod_{i=1}^n [y_i-2^{N_0}, y_i+2^{N_0}]} \left| \delta^{-n} \int_{|x-y|<\delta} \omega\left(\frac{x-y}{\delta}\right) [u(x)-u(y)] dy \right|^p dx \right]^{1/p} \\ &\lesssim \tilde{\varepsilon} 2^{-N_1 n(\frac{1}{u}-\frac{1}{p})} 2^{N_0 n/p} \lesssim \varepsilon, \end{aligned}$$

if δ is chosen small enough, where $u^{(\delta)}$ is defined as in (4.9) with ϕ and ε replaced, respectively, by u and δ . This, together with (4.20), (4.21), (4.23) and the definition of $u^{(\delta)}$, further implies that

$$\|u - u^{(\delta)}\|_{\mathcal{M}_p^u(\mathbb{R}^n)} \lesssim \varepsilon \quad (4.24)$$

if δ is sufficiently small. Since $u^{(\delta)}$ is smooth, we obtain the desired conclusion in Substep 1.2.

Substep 1.3. The final step consists now in approximating $u^{(\delta)}$ by compactly supported smooth functions. Let ψ be as in (5.1). We define

$$u_\ell^{(\delta)}(x) := \psi(x/\ell) u^{(\delta)}(x), \quad x \in \mathbb{R}^n, \quad \ell \in \mathbb{N}.$$

Of course, $u^{(\delta)}$ and $u_\ell^{(\delta)}$ have the properties (2.10), (2.11) and (2.12), since $0 \leq \psi(x) \leq 1$ for all x . Similar to the above estimates (4.20) and (4.21) for g , we conclude that, for given $\varepsilon \in (0, \infty)$, there exists $\varepsilon_1 \in (0, \infty)$ such that

$$\sup_{|B|<\varepsilon_1} |B|^{\frac{1}{u}-\frac{1}{p}} \left[\int_B |u^{(\delta)}(x)|^p dx \right]^{1/p} \leq \varepsilon \quad (4.25)$$

and

$$\sup_{1/\varepsilon_1 < |B|} |B|^{\frac{1}{u}-\frac{1}{p}} \left[\int_B |u^{(\delta)}(x)|^p dx \right]^{1/p} \leq \varepsilon. \quad (4.26)$$

In addition, since $u^{(\delta)}$ satisfies (2.12), we know that there exists $\varepsilon_2 \in (0, \infty)$ such that

$$\sup_{|y|>1/\varepsilon_2} |B(y, r)|^{\frac{1}{u}-\frac{1}{p}} \left[\int_{B(y, r)} |u^{(\delta)}(x)|^p dx \right]^{1/p} \leq \varepsilon. \quad (4.27)$$

All three inequalities above remain true with $u^{(\delta)}$ replaced by $u_\ell^{(\delta)}$ since $0 \leq \psi \leq 1$. From these observations (4.25), (4.26) and (4.27), we deduce that

$$\begin{aligned} &\|u^{(\delta)} - u_\ell^{(\delta)}\|_{\mathcal{M}_p^u(\mathbb{R}^n)} \\ &\leq \sup_{|y| \leq 1/\varepsilon_2} \sup_{\varepsilon_1 < |B(y, r)| < 1/\varepsilon_1} |B(y, r)|^{\frac{1}{u}-\frac{1}{p}} \left[\int_{B(y, r)} |u^{(\delta)}(x) - u_\ell^{(\delta)}(x)|^p dx \right]^{1/p} \\ &\quad + \sup_{\substack{|y| > 1/\varepsilon_2 \\ r \in (0, \infty)}} \cdots + \sup_{\substack{y \in \mathbb{R}^n \\ |B(y, r)| \geq 1/\varepsilon_1}} \cdots + \sup_{\substack{y \in \mathbb{R}^n \\ |B(y, r)| \leq \varepsilon_1}} \cdots \\ &\lesssim \sup_{|y| \leq 1/\varepsilon_2} \sup_{\varepsilon_1 < |B(y, r)| < 1/\varepsilon_1} |B(y, r)|^{\frac{1}{u}-\frac{1}{p}} \left[\int_{B(y, r)} |u^{(\delta)}(x) - u_\ell^{(\delta)}(x)|^p dx \right]^{1/p} + 6\varepsilon. \end{aligned} \quad (4.28)$$

Choosing

$$\ell \geq \frac{1}{\varepsilon_2} + \left(\frac{1}{\varepsilon_1} \right)^{1/n}, \quad (4.29)$$

we find that the first term on the right-hand side of (4.28) vanishes, due to the definition of $u_\ell^{(\delta)}$. Combining this observation, (4.23) and (4.24), we know that $\|g - u_\ell^{(\delta)}\|_{\mathcal{M}_p^u(\mathbb{R}^n)} \lesssim \varepsilon$ if ℓ is large enough and δ is small enough, namely, g can be approximated by a sequence of smooth compactly supported functions in $\mathcal{M}_p^u(\mathbb{R}^n)$. Hence $M(\mathbb{R}^n) = \dot{\mathcal{M}}_p^u(\mathbb{R}^n)$. This proves (i).

Step 2. Proof of (ii). We proceed as in Step 1. This time $M(\mathbb{R}^n)$ denotes the space of all functions $g \in \mathcal{M}_p^u(\mathbb{R}^n)$ having the properties (2.11) (uniformly in $y \in \mathbb{R}^n$) and (2.12) (uniformly in r).

First we prove $\dot{\mathcal{M}}_p^u(\mathbb{R}^n) \hookrightarrow M(\mathbb{R}^n)$. It is easy to see that, if f is a compactly supported function, then f satisfies (2.11) and (2.12), due to the compactness of its support. Moreover, the limits of compactly supported functions in $\mathcal{M}_p^u(\mathbb{R}^n)$ also satisfy these two properties (2.11) and (2.12). This proves $\dot{\mathcal{M}}_p^u(\mathbb{R}^n) \hookrightarrow M(\mathbb{R}^n)$.

Next we show $M(\mathbb{R}^n) \hookrightarrow \dot{\mathcal{M}}_p^u(\mathbb{R}^n)$. Let $f \in M(\mathbb{R}^n)$. We need to find a sequence of compactly supported functions which converges to f in $\mathcal{M}_p^u(\mathbb{R}^n)$. Indeed, the desired approximating sequence of f in $\mathcal{M}_p^u(\mathbb{R}^n)$ is simply given by

$$f_\ell(x) := f(x) \chi_{B(0, \ell)}(x), \quad x \in \mathbb{R}^n, \quad \ell \in \mathbb{N}.$$

To see this, since f and $\{f_\ell\}_{\ell \in \mathbb{Z}_+}$ are all elements of $M(\mathbb{R}^n)$, we conclude that (4.26) and (4.27) remain true for f and f_ℓ with $\ell \in \mathbb{Z}_+$. Then, similar to the proof of (4.28), we see that

$$\|f^{(\delta)} - f_\ell^{(\delta)}\|_{\mathcal{M}_p^u(\mathbb{R}^n)} \lesssim \varepsilon + \sup_{|y| \leq 1/\varepsilon_2} \sup_{|B(y, r)| < 1/\varepsilon_1} |B(y, r)|^{\frac{1}{u} - \frac{1}{p}} \left[\int_{B(y, r)} |f^{(\delta)}(x) - f_\ell^{(\delta)}(x)|^p dx \right]^{1/p}.$$

Choosing ℓ as in (4.29), we find that the second term on the right-hand side of the above inequality vanishes. On the other hand, similar to the arguments used in Substep 1.2, we know that

$$\|f - f^{(\delta)}\|_{\mathcal{M}_p^u(\mathbb{R}^n)} + \|f_\ell - f_\ell^{(\delta)}\|_{\mathcal{M}_p^u(\mathbb{R}^n)} \lesssim \varepsilon$$

if δ is small enough. Altogether we find that f can be approximated by f_ℓ in $\mathcal{M}_p^u(\mathbb{R}^n)$. This proves $M(\mathbb{R}^n) = \dot{\mathcal{M}}_p^u(\mathbb{R}^n)$.

Step 3. Proof of (iii). This time we let $M(\mathbb{R}^n)$ denote the collection of all $f \in \mathcal{M}_p^u(\mathbb{R}^n)$ such that (2.10) holds true uniformly in $y \in \mathbb{R}^n$.

First we show $\dot{\mathcal{M}}_p^u(\mathbb{R}^n) \hookrightarrow M(\mathbb{R}^n)$. Let f be a $C^\infty(\mathbb{R}^n)$ function such that f and all its derivatives $D^\alpha f$ belong to $\mathcal{M}_p^u(\mathbb{R}^n)$. From this and Proposition 5.2(iv), it follows that $f \in W^m \mathcal{M}_p^u(\mathbb{R}^n)$ for any $m \in \mathbb{N}$, as well as all of its derivatives, here $W^m \mathcal{M}_p^u(\mathbb{R}^n)$ denotes the Morrey-Sobolev space of order m . Since $W^m \mathcal{M}_p^u(\mathbb{R}^n) \hookrightarrow L_\infty(\mathbb{R}^n)$ for sufficiently large m , we know that f and its derivatives $D^\alpha f$ are all bounded. Consequently, we conclude that

$$\begin{aligned} |B(y, r)|^{\frac{1}{u} - \frac{1}{p}} \left[\int_{B(y, r)} |f(x)|^p dx \right]^{1/p} &\leq |B(y, r)|^{\frac{1}{u} - \frac{1}{p}} \left[\int_{B(y, r)} |f(y)|^p dx \right]^{1/p} \\ &\quad + |B(y, r)|^{\frac{1}{u} - \frac{1}{p}} \left[\int_{B(y, r)} |f(x) - f(y)|^p dx \right]^{1/p} \\ &\lesssim |f(y)| |B(y, r)|^{\frac{1}{u}} + \max_{|\alpha|=1} \|D^\alpha f\|_{L_\infty(\mathbb{R}^n)} |B(y, r)|^{\frac{1}{u} + \frac{1}{n}} \\ &\lesssim \max_{|\alpha| \leq 1} \|D^\alpha f\|_{L_\infty(\mathbb{R}^n)} |B(y, r)|^{\frac{1}{u}} \max\{r, 1\}. \end{aligned}$$

Clearly, the right-hand side of the above inequalities tends to 0 if $r \downarrow 0$ (uniformly in y). This property carries over to the limits in $\mathcal{M}_p^u(\mathbb{R}^n)$ of such kind of functions. Hence $\dot{\mathcal{M}}_p^u(\mathbb{R}^n) \hookrightarrow M(\mathbb{R}^n)$.

It remains to prove that any $f \in M(\mathbb{R}^n)$ can be approximated by functions which, together with all its derivatives, belong to $\mathcal{M}_p^u(\mathbb{R}^n)$. We shall work with the Sobolev mollification $f^{(\delta)}$ of f . By the definition of the Sobolev mollification and the generalized Minkowski inequality, we have

$$\begin{aligned} \left[\int_{B(y, r)} |D^\alpha f^{(\delta)}(x)|^p dx \right]^{1/p} &= \left[\int_{B(y, r)} \left| \delta^{-n-|\alpha|} \int_{\mathbb{R}^n} (D^\alpha \omega) \left(\frac{x-y}{\delta} \right) f(y) dy \right|^p dx \right]^{1/p} \\ &= \left[\int_{B(y, r)} \left| \delta^{-n-|\alpha|} \int_{\mathbb{R}^n} (D^\alpha \omega) \left(\frac{z}{\delta} \right) f(x-z) dz \right|^p dx \right]^{1/p} \end{aligned}$$

$$\begin{aligned}
&\leq \delta^{-n-|\alpha|} \int_{\mathbb{R}^n} \left| (D^\alpha \omega) \left(\frac{z}{\delta} \right) \right| \left[\int_{B(y,r)} |f(x-z)|^p dx \right]^{1/p} dz \\
&\leq \delta^{-|\alpha|} \sup_{z \in \mathbb{R}^n} \left[\int_{B(z,r)} |f(x)|^p dx \right]^{1/p}.
\end{aligned} \tag{4.30}$$

This shows that also $f^{(\delta)}$ and all its derivatives $D^\alpha f^{(\delta)}$ belong to $\mathcal{M}_p^u(\mathbb{R}^n)$. Since f satisfies (2.10), we see that, for given $\varepsilon \in (0, \infty)$, there exists $\varepsilon_1 \in (0, \infty)$ such that

$$\sup_{|B| < \varepsilon_1} |B|^{\frac{1}{u} - \frac{1}{p}} \left[\int_B |f(x)|^p dx \right]^{1/p} \leq \varepsilon.$$

Hence, employing (4.30) with $\alpha = 0$ and the Minkowski inequality, we find that

$$\begin{aligned}
\|f^{(\delta)} - f\|_{\mathcal{M}_p^u(\mathbb{R}^n)} &\leq 2\varepsilon + \sup_{|B| \geq \varepsilon_1} |B|^{\frac{1}{u} - \frac{1}{p}} \left[\int_B |f(x) - f^{(\delta)}(x)|^p dx \right]^{1/p} \\
&\leq 2\varepsilon + \delta^{-n} \sup_{|B| \geq \varepsilon_1} |B|^{\frac{1}{u} - \frac{1}{p}} \left\{ \int_B \left| \int_{\mathbb{R}^n} \omega \left(\frac{x-y}{\delta} \right) [f(x) - f(y)] dy \right|^p dx \right\}^{1/p} \\
&\leq 2\varepsilon + \sup_{|B| \geq \varepsilon_1} |B|^{\frac{1}{u} - \frac{1}{p}} \sup_{|h| \leq \delta} \left[\int_B |f(x) - f(x+h)|^p dx \right]^{1/p}.
\end{aligned}$$

By the definition of the supremum, there exists a sequence $\{(y_j, r_j)\}_{j \in \mathbb{N}}$ such that

$$\begin{aligned}
&\sup_{|B| \geq \varepsilon_1} \sup_{|h| \leq \delta} |B|^{\frac{1}{u} - \frac{1}{p}} \left[\int_B |f(x) - f(x+h)|^p dx \right]^{1/p} \\
&< \frac{1}{j} + \sup_{|h| \leq \delta} |B(y_j, r_j)|^{\frac{1}{u} - \frac{1}{p}} \left[\int_{B(y_j, r_j)} |f(x) - f(x+h)|^p dx \right]^{1/p} \leq 2\varepsilon,
\end{aligned}$$

if j is large enough and if δ is small enough (since, for a fixed j , we can apply the $L_p(\mathbb{R}^n)$ -continuity of the translation). Inserting this inequality into the previous one, we are done, which completes the proof of Lemma 2.33.

Proof of Lemma 2.35

Part (i) is already proved by using the example (2.13). Part (ii) follows from Lemma 2.33. We focus on (iii). Obviously, the function $g_{n/u}$ in (2.14) belongs to $\hat{\mathcal{M}}_p^u(\mathbb{R}^n)$. However, it does not belong to $\hat{\mathcal{M}}_p^u(\mathbb{R}^n)$, since, according to Lemma 2.33(iii), functions from this space satisfy (2.10) but

$$\lim_{r \downarrow 0} |B(0, r)|^{\frac{1}{u} - \frac{1}{p}} \left[\int_{B(0, r)} |x|^{np/u} dx \right] > 0,$$

which implies that $g_{n/u}$ does not satisfy (2.10). Moreover, the function $h_{n/u}$ in (2.15) belongs to $\hat{\mathcal{M}}_p^u(\mathbb{R}^n)$, since $h_{n/u}$ is a $C^\infty(\mathbb{R}^n)$ function such that all derivatives also belong to $\mathcal{M}_p^u(\mathbb{R}^n)$. It does not belong to $\hat{\mathcal{M}}_p^u(\mathbb{R}^n)$, since, according to Lemma 2.33(iii), functions from this space satisfy (2.11) but

$$\lim_{r \downarrow \infty} |B(0, r)|^{\frac{1}{u} - \frac{1}{p}} \left[\int_{B(0, r)} |x|^{np/u} dx \right] > 0$$

which implies that $h_{n/u}$ does not satisfy (2.11). This proves (iii) and hence finishes the proof of Lemma 2.35.

Proof of Lemma 2.37

Step 1. Preliminaries. For any ball $B \subset \mathbb{R}^n$, the Hölder inequality yields

$$\left[\int_B |f(x)|^p dx \right]^{1/p} \leq \left[\int_B |f(x)|^{p_0} dx \right]^{(1-\Theta)/p_0} \left[\int_B |f(x)|^{p_1} dx \right]^{\Theta/p_1},$$

which implies that

$$|B|^{\frac{1}{u}-\frac{1}{p}} \left[\int_B |f(x)|^p dx \right]^{1/p} \leq \|f\|_{\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n)}^{1-\Theta} \|f\|_{\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)}^{\Theta}.$$

In other words,

$$\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \hookrightarrow \mathcal{M}_p^u(\mathbb{R}^n).$$

Step 2. Suppose $f \in \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$. By $p_0 \leq p \leq p_1$ and the Hölder inequality, we know that

$$\begin{aligned} |B(y, r)|^{\frac{1}{u}-\frac{1}{p}} \left[\int_{B(y, r)} |f(x)|^p dx \right]^{1/p} &\leq |B(y, r)|^{\frac{1}{u}-\frac{1}{p_1}} \left[\int_{B(y, r)} |f(x)|^{p_1} dx \right]^{1/p_1} \\ &= |B(y, r)|^{\frac{1}{u}-\frac{1}{u_1}} |B(y, r)|^{\frac{1}{u_1}-\frac{1}{p_1}} \left[\int_{B(y, r)} |f(x)|^{p_1} dx \right]^{1/p_1} \\ &\leq |B(y, r)|^{\frac{1}{u}-\frac{1}{u_1}} \|f\|_{\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)}, \end{aligned} \quad (4.31)$$

which converges to 0 as $r \rightarrow 0$, due to $u_1 > u$. By Lemma 2.33(iii), this proves $f \in \mathring{\mathcal{M}}_p^u(\mathbb{R}^n)$ if $p \in [1, \infty)$. Thus, if $p \in [1, \infty)$,

$$\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \hookrightarrow \mathring{\mathcal{M}}_p^u(\mathbb{R}^n).$$

Now we argue by using our test function g_α in (2.14) with $\alpha = n/u$ to show that $\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$ is not dense in $\mathcal{M}_p^u(\mathbb{R}^n)$. It is known that this time $g_\alpha \in \mathcal{M}_p^u(\mathbb{R}^n)$. From $p < u$ and (4.31), it follows that

$$\lim_{r \downarrow 0} |B(0, r)|^{\frac{1}{u}-\frac{1}{p}} \left[\int_{B(0, r)} |g_\alpha(x) - f(x)|^p dx \right]^{1/p} = \lim_{r \downarrow 0} |B(0, r)|^{\frac{1}{u}-\frac{1}{p}} \left[\int_{B(0, r)} |g_\alpha(x)|^p dx \right]^{1/p} > 0$$

for all $f \in \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$. This means that there exist functions in $\mathcal{M}_p^u(\mathbb{R}^n)$ that can not be approximated by functions from $\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$, namely, $\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$ is not dense in $\mathcal{M}_p^u(\mathbb{R}^n)$, which completes the proof of Lemma 2.37.

Remark 4.12. In the above proof of Lemma 2.37, we prove more than stated. Indeed, we show

$$(i) \quad \mathcal{M}_p^u(\mathbb{R}^n) \not\subset \overline{\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)}^{\|\cdot\|_{\mathcal{M}_p^u(\mathbb{R}^n)}},$$

but

$$(ii) \quad \overline{\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)}^{\|\cdot\|_{\mathcal{M}_p^u(\mathbb{R}^n)}} \hookrightarrow \mathring{\mathcal{M}}_p^u(\mathbb{R}^n).$$

Proof of Corollary 2.38

Step 1. Proof of (i). Theorem 2.5(i) and Proposition 4.3 yield

$$\langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_{\Theta} \hookrightarrow [\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n)]^{1-\Theta} [\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)]^{\Theta} \hookrightarrow \mathcal{M}_p^u(\mathbb{R}^n).$$

Theorem 2.5(iii) implies that this embedding is proper if $u_0 p_1 \neq u_1 p_0$.

It remains to consider the case $u_0 p_1 = u_1 p_0$. Without loss of generality, we may assume $p_0 \leq p_1$. Under these conditions, by Proposition 2.20 and Corollary 2.14(i), we see that

$$\langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_{\Theta} = \overline{\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)}^{\|\cdot\|_{\mathcal{M}_p^u(\mathbb{R}^n)}}. \quad (4.32)$$

If $p_0 = u_0$ and $p_1 = u_1$, then we have

$$\langle \mathcal{M}_{p_0}^{p_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{p_1}(\mathbb{R}^n) \rangle_{\Theta} = \overline{L_{p_0}(\mathbb{R}^n) \cap L_{p_1}(\mathbb{R}^n)}^{\|\cdot\|_{L_p(\mathbb{R}^n)}} = L_p(\mathbb{R}^n) = \mathcal{M}_p^p(\mathbb{R}^n).$$

If $p_0 = p_1$, we must have $u_0 = u_1$ and therefore

$$\langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_{\Theta} = \langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \rangle_{\Theta} = \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n).$$

If $p_0 \neq u_0$ and $p_0 < p_1$, then $u_0 < u_1$ follows and therefore $u < u_1$. In this case, Lemma 2.37 yields the desired conclusion. This proves (i).

Step 2. Proof of (ii). By Proposition 2.20 and Corollary 2.14(i), it is known that

$$\dot{\mathcal{M}}_p^u(\mathbb{R}^n) \hookrightarrow \langle \dot{\mathcal{M}}_{p_0}^{u_0}(\mathbb{R}^n), \dot{\mathcal{M}}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_{\Theta} \hookrightarrow \langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_{\Theta}.$$

On the other hand, it is easy to show that the test function g_{α} in (2.14) with $\alpha = n/u_1$ belongs to $\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$, and hence belongs to $\langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_{\Theta}$. However, Step 2 of the proof of Lemma 2.37 implies that g_{α} can not be approximated by $C_c^{\infty}(\mathbb{R}^n)$ functions in the norm $\|\cdot\|_{\mathcal{M}_p^u(\mathbb{R}^n)}$, due to Lemma 2.33(i). Namely, g_{α} does not belong to $\dot{\mathcal{M}}_p^u(\mathbb{R}^n)$. This proves (ii).

Step 3. Proof of (iii). Since $p_0 < p_1$, it follows $u_0 < u_1$. Hence, we apply Lemma 2.37 in case $p \in [1, \infty)$ and conclude

$$\langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_{\Theta} \hookrightarrow \dot{\mathcal{M}}_p^u(\mathbb{R}^n).$$

It remains to show that these spaces do not coincide. This time we argue with our test function $h_{n/u}$ in (2.15). Clearly, $h_{n/u} \in \dot{\mathcal{M}}_p^u(\mathbb{R}^n)$. We now show that

$$h_{n/u} \notin \langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_{\Theta}.$$

By (4.32), we only need to show that

$$h_{n/u} \notin \overline{\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)}^{\|\cdot\|_{\mathcal{M}_p^u(\mathbb{R}^n)}}.$$

Observe that, by an elementary calculation, we know that, for all $t \in (0, \infty)$,

$$2^{jn(\frac{1}{u}-\frac{1}{t})} \left[\int_{2^j \leq |x| \leq 2^{j+1}} |h_{n/u}(x)|^t dx \right]^{1/t} = C_{(n,t,u)} > 0, \quad j \in \mathbb{N},$$

where $C_{(n,t,u)}$ denotes a positive constant depending on n , t and u . Now, for any given $\varepsilon \in (0, \infty)$, assume that there exists $f_{\varepsilon} \in \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$ such that $\|h_{n/u} - f_{\varepsilon}\|_{\mathcal{M}_p^u(\mathbb{R}^n)} < \varepsilon$. Without loss of generality, we may assume $|f_{\varepsilon}(x)| \leq |x|^{-n/u}$ for all $x \in \mathbb{R}^n \setminus \{(0, \dots, 0)\}$ (otherwise, we switch to $\min\{|f_{\varepsilon}(x)|, |x|^{-n/u}\}$). Fixing $\varepsilon \in (0, \infty)$ sufficiently small, we conclude that

$$2^{jn(\frac{1}{u}-\frac{1}{p})} \left[\int_{2^j \leq |x| \leq 2^{j+1}} |f_{\varepsilon}(x)|^p dx \right]^{1/p} > \frac{C_{(n,p,u)}}{2}, \quad j \in \mathbb{N}, \quad (4.33)$$

where $C_{(n,p,u)}$ denotes a positive constant depending on n , p and u . But this is in contradiction with $f_{\varepsilon} \in \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$. To explain this contradiction, by our pointwise assumption $|f_{\varepsilon}(x)| \leq |x|^{-n/u}$ for all $x \in \mathbb{R}^n \setminus \{(0, \dots, 0)\}$, we see that

$$\left[\int_{2^j \leq |x| \leq 2^{j+1}} |f_{\varepsilon}(x)|^{p_1} dx \right]^{1/p_1} \lesssim 2^{jn(\frac{1}{p_1}-\frac{1}{u})}, \quad j \in \mathbb{N}. \quad (4.34)$$

Finally, we employ the Hölder inequality, (4.33) and (4.34) to find that

$$\frac{C_{(n,p,u)}}{2} < 2^{jn(\frac{1}{u}-\frac{1}{p})} \left[\int_{2^j \leq |x| \leq 2^{j+1}} |f_{\varepsilon}(x)|^p dx \right]^{1/p}$$

$$\begin{aligned}
&\leq \left\{ 2^{jn(\frac{1}{u_0} - \frac{1}{p_0})} \left[\int_{2^j \leq |x| \leq 2^{j+1}} |f_\varepsilon(x)|^{p_0} dx \right]^{1/p_0} \right\}^{1-\Theta} \\
&\quad \times \left\{ 2^{jn(\frac{1}{u_1} - \frac{1}{p_1})} \left[\int_{2^j \leq |x| \leq 2^{j+1}} |f_\varepsilon(x)|^{p_1} dx \right]^{1/p_1} \right\}^\Theta \\
&\lesssim \|f_\varepsilon\|_{\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n)}^{1-\Theta} 2^{jn(\frac{1}{u_1} - \frac{1}{u})\Theta}, \quad j \in \mathbb{N}.
\end{aligned}$$

Because of $u < u_1$, the right-hand side of the above inequalities tends to zero for j tending to infinity. This is a contradiction. Hence, our assumption, that there is a function $f_\varepsilon \in \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$ in an ε distance of h_n/u , is impossible. This proves

$$\langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_\Theta \subsetneq \dot{\mathcal{M}}_p^u(\mathbb{R}^n),$$

which completes the proof of Corollary 2.38.

Proof of Theorem 2.40

Step 1. We prove

$$\langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_\Theta \hookrightarrow \mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n).$$

Let $f \in \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$. Then $f \in \dot{\mathcal{M}}_p^u(\mathbb{R}^n)$ follows from Lemma 2.37, which further implies the validity of (2.19) and (2.20) due to Lemma 2.33(iii). The conditions (2.21) and (2.22) are obviously true since $f \in \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$. Thus, we conclude that $f \in \mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n)$, which implies that

$$\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \hookrightarrow \mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n).$$

Taking the closure in $\mathcal{M}_p^u(\mathbb{R}^n)$, by Proposition 2.20 and Corollary 2.14(i), we obtain

$$\langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_\Theta \hookrightarrow \mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n).$$

Step 2. It remains to show

$$\mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n) \hookrightarrow \langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_\Theta.$$

We claim

$$\mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n) = \dot{\mathcal{M}}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n). \quad (4.35)$$

To prove this claim, we need $p_0 \in [1, \infty)$. Clearly, we work again with the Sobolev mollification. As in Step 3 of the proof of Lemma 2.33, it follows that, for any $f \in \mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n)$,

$$I_1(f - f^{(\delta)}) + I_2(f - f^{(\delta)}) + I_3(f - f^{(\delta)}) \rightarrow 0$$

if $\delta \downarrow 0$, which implies $f \in \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$ and hence

$$\mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n) \hookrightarrow \dot{\mathcal{M}}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n). \quad (4.36)$$

Now, let $f \in \dot{\mathcal{M}}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n)$, namely, $f \in \mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n)$ such that all derivatives $D^\alpha f$, $\alpha \in \mathbb{Z}_+^n$, belong to $\mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n)$ as well. By the definition of $\mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n)$ and the Hölder inequality, we obtain $D^\alpha f \in \mathcal{M}_p^u(\mathbb{R}^n)$ for all $\alpha \in \mathbb{Z}_+^n$. Since $W^m(\mathcal{M}_p^u(\mathbb{R}^n)) \hookrightarrow L_\infty(\mathbb{R}^n)$ if $m > n/p$ (see [21]), we conclude $f \in L_\infty(\mathbb{R}^n)$. By Definition 2.39, it is easy to see that

$$L_\infty(\mathbb{R}^n) \cap \mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n) \hookrightarrow \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n).$$

Therefore, by this, Proposition 2.20 and Step 1, we further conclude that

$$\dot{\mathcal{M}}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n) \hookrightarrow \overline{\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)}^{\|\cdot\|_{\mathcal{M}_p^u(\mathbb{R}^n)}} = \langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_\Theta$$

$$\hookrightarrow \mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n), \quad (4.37)$$

since the convergence in $\|\cdot\|_{\mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n)}$ implies the convergence in $\|\cdot\|_{\mathcal{M}_p^u(\mathbb{R}^n)}$ by the Hölder inequality. Combining (4.36) and (4.37), we obtain (4.35), which, together with (4.37) again, implies that

$$\langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_{\Theta} = \mathcal{M}_{p_0, p_1}^{u_0, u_1, \Theta}(\mathbb{R}^n).$$

This finishes the proof of Theorem 2.40.

Proof of Theorem 2.44

First we have to prove the formula

$$\langle \mathcal{M}_{p_0}^{u_0}((0, 1)^n), \mathcal{M}_{p_1}^{u_1}((0, 1)^n), \Theta \rangle = \mathcal{M}_p^u((0, 1)^n). \quad (4.38)$$

This can be done by the method of the retraction and the coretraction. Here the coretraction is defined as the extension from $(0, 1)^n$ to \mathbb{R}^n by zero. Hence, (4.38) becomes a consequence of Corollary 2.14(i). Second, we employ Proposition 2.20 to obtain

$$\langle \mathcal{M}_{p_0}^{u_0}((0, 1)^n), \mathcal{M}_{p_1}^{u_1}((0, 1)^n) \rangle_{\Theta} = \overline{\mathcal{M}_{p_0}^{u_0}((0, 1)^n) \cap \mathcal{M}_{p_1}^{u_1}((0, 1)^n)}^{\|\cdot\|_{\mathcal{M}_p^u((0, 1)^n)}}.$$

Next we continue, as in Lemma 2.37, to conclude that

$$\mathcal{M}_{p_0}^{u_0}((0, 1)^n) \cap \mathcal{M}_{p_1}^{u_1}((0, 1)^n) \hookrightarrow \mathring{\mathcal{M}}_p^u((0, 1)^n).$$

Finally, we observe

$$\begin{aligned} & \left\{ f \in C^\infty((0, 1)^n) : D^\alpha f \in L_\infty((0, 1)^n) \text{ for all } \alpha \in \mathbb{Z}_+^n \right\} \\ & \subset \left(\mathcal{M}_{p_0}^{u_0}((0, 1)^n) \cap \mathcal{M}_{p_1}^{u_1}((0, 1)^n) \right) \subset \mathring{\mathcal{M}}_p^u((0, 1)^n). \end{aligned}$$

Taking the closure in both sides of the above formula with respect to the norm $\|\cdot\|_{\mathcal{M}_p^u(\mathbb{R}^n)}$ (see Lemma 2.43), we then obtain the desired conclusion of Theorem 2.44.

Proof of Theorem 2.45

To prove Theorem 2.45, we need the following conclusions, which have their own interest. The first one shows that, if Ω is a Lipschitz domain, then there exists a universal linear bounded extension operator from $\mathcal{N}_{u, p, q}^s(\Omega)$ into $\mathcal{N}_{u, p, q}^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, $q \in (0, \infty]$ and $0 < p \leq u < \infty$. In the construction of this operator we follow Rychkov [74, Theorem 2.2].

Proposition 4.13. *Let $\Omega \subset \mathbb{R}^n$ be an interval if $n = 1$ or a Lipschitz domain if $n \geq 2$. Then there exists a linear bounded operator \mathcal{E} which maps $\mathcal{N}_{u, p, q}^s(\Omega)$ into $\mathcal{N}_{u, p, q}^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, $q \in (0, \infty]$ and $0 < p \leq u < \infty$ such that, for all $f \in D'(\Omega)$, $\mathcal{E}f|_\Omega = f$ in $D'(\Omega)$.*

Proof. By similarity, we concentrate us on the case $n \geq 2$. A standard procedure (see, for example, [74, Subsection 1.2]) shows that, to prove Proposition 4.13, we only need to consider the case when Ω is a special Lipschitz domain. In this case, let

$$K := \{(x', x_n) \in \mathbb{R}^n : |x'| < A^{-1}x_n\}$$

and $-K := \{-x : x \in K\}$, where A is the Lipschitz constant of the boundary Lipschitz function ω of Ω . Then K has the property that $x + K \subset \Omega$ for any $x \in \Omega$.

Let $\phi_0 \in D(-K)$ and $\phi(\cdot) := \phi_0(\cdot) - \phi_0(\cdot/2)$ be such that $\int_{\mathbb{R}^n} \phi_0(x) dx \neq 0$ and $L_\phi \geq \lfloor s \rfloor$. Here and hereafter, L_ϕ denotes the maximal number such that $\int_{\mathbb{R}^n} \phi(x) x^\alpha dx = 0$ for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq L_\phi$.

Then, by [74, Proposition 2.1], there exist functions ψ_0 and ψ in $D(-K)$ such that $L_\psi \geq L_\phi$ and, for all $f \in D'(\Omega)$,

$$f = \sum_{j \in \mathbb{Z}_+} \psi_j * \phi_j * f$$

in $D'(\Omega)$. For all $f \in D'(\Omega)$, we define

$$\mathcal{E}f := \sum_{j \in \mathbb{Z}_+} \psi_j * (\phi_j * f)_\Omega, \quad (4.39)$$

here and hereafter, for any function $g : \Omega \rightarrow \mathbb{R}$, g_Ω denotes the extension of g from Ω to \mathbb{R}^n by setting $g_\Omega(x) := g(x)$ if $x \in \Omega$ and $g_\Omega(x) := 0$ if $x \in \mathbb{R}^n \setminus \Omega$.

For all $s \in \mathbb{R}$, $q \in (0, \infty]$ and $0 < p \leq u < \infty$, let $\ell_q^s(\mathcal{M}_p^u(\mathbb{R}^n))$ be the space of all sequences $\{g_j\}_{j \in \mathbb{Z}_+}$ of measurable functions on \mathbb{R}^n such that

$$\|\{g_j\}_{j \in \mathbb{Z}_+}\|_{\ell_q^s(\mathcal{M}_p^u(\mathbb{R}^n))} := \left\{ \sum_{j \in \mathbb{Z}_+} 2^{jsq} \|G_j\|_{\mathcal{M}_p^u(\mathbb{R}^n)}^q \right\}^{1/q} < \infty,$$

where G_j denotes the Peetre maximal function of g_j , namely,

$$G_j(x) := \sup_{y \in \mathbb{R}^n} \frac{|g_j(y)|}{(1 + 2^j|x - y|)^N}$$

for all $x \in \mathbb{R}^n$ and $N \in \mathbb{N} \cap (\frac{n}{\min\{1, p\}}, \infty)$. By [74, (2.14)], we know that, if $L_\phi \geq \lfloor s \rfloor$ and $L_\psi \geq N$, then there exists $\sigma \in (0, \infty)$ such that, for any sequence $\{g_j\}_{j \in \mathbb{Z}_+}$ with $\|\{g_j\}_{j \in \mathbb{Z}_+}\|_{\ell_q^s(\mathcal{M}_p^u(\mathbb{R}^n))} < \infty$, it holds true that

$$2^{ls} |\phi_l * \psi_j * g_j(x)| \lesssim 2^{-|l-j|\sigma} 2^{js} G_j(x), \quad x \in \mathbb{R}^n, \quad l \in \mathbb{Z}_+,$$

and hence

$$\|\psi_j * g_j\|_{\mathcal{N}_{u,p,q}^{s-2\sigma}(\mathbb{R}^n)} \lesssim \left[\sum_{l \in \mathbb{Z}_+} 2^{l(-2\sigma+|l-j|\sigma)q} \right]^{1/q} \|2^{js} G_j\|_{\mathcal{M}_p^u(\mathbb{R}^n)} \lesssim 2^{-j\sigma} \|\{g_j\}_{j \in \mathbb{Z}_+}\|_{\ell_q^s(\mathcal{M}_p^u(\mathbb{R}^n))}.$$

This implies that $\sum_{j \in \mathbb{Z}_+} \psi_j * g_j$ converges in $\mathcal{N}_{u,p,q}^{s-2\sigma}(\mathbb{R}^n)$ and hence in $\mathcal{S}'(\mathbb{R}^n)$, since $\mathcal{N}_{u,p,q}^{s-2\sigma}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. Therefore, we further have

$$2^{ls} \left| \phi_l * \left(\sum_{j \in \mathbb{Z}_+} \psi_j * g_j \right) (x) \right| \lesssim \sum_{j \in \mathbb{Z}_+} 2^{-|l-j|\sigma} 2^{js} G_j(x), \quad x \in \mathbb{R}^n, \quad l \in \mathbb{Z}_+.$$

Applying this, we then see that

$$\left\| \sum_{j \in \mathbb{Z}_+} \psi_j * g_j \right\|_{\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)} \lesssim \|\{g_j\}_{j \in \mathbb{Z}_+}\|_{\ell_q^s(\mathcal{M}_p^u(\mathbb{R}^n))}. \quad (4.40)$$

Let $f \in \mathcal{N}_{u,p,q}^s(\Omega)$. Then, for any $\varepsilon \in (0, \infty)$, there exists $g \in \mathcal{N}_{u,p,q}^s(\mathbb{R}^n)$ such that $g|_\Omega = f$ in $D'(\Omega)$ and

$$\|g\|_{\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)} \leq \|f\|_{\mathcal{N}_{u,p,q}^s(\Omega)} + \varepsilon.$$

Let $g_j := (\phi_j * f)_\Omega$ for all $j \in \mathbb{Z}_+$. By (4.40) and the fact (see [74, pp. 247-248]) that

$$\sup_{y \in \Omega} \frac{|\phi_j * f(y)|}{(1 + 2^j|x - y|)^N} \begin{cases} = \sup_{y \in \Omega} \frac{|\phi_j * f(y)|}{(1 + 2^j|x - y|)^N}, & x \in \Omega, \\ \lesssim \sup_{y \in \Omega} \frac{|\phi_j * f(y)|}{(1 + 2^j|\tilde{x} - y|)^N}, & x \notin \overline{\Omega}, \end{cases}$$

$$\begin{cases} \leq \sup_{y \in \mathbb{R}^n} \frac{|\phi_j * g(y)|}{(1 + 2^j|x - y|)^N}, & x \in \Omega, \\ \lesssim \sup_{y \in \mathbb{R}^n} \frac{|\phi_j * g(y)|}{(1 + 2^j|\tilde{x} - y|)^N}, & x \notin \overline{\Omega}, \end{cases}$$

where $\tilde{x} := (x', 2w(x') - x_n) \in \Omega$ is the symmetric point to $x = (x', x_n) \notin \overline{\Omega}$ with respect to $\partial\Omega$, we conclude that

$$\begin{aligned} \|\mathcal{E}f\|_{\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)} &\lesssim \|\{(\phi_j * f)_\Omega\}_{j \in \mathbb{Z}_+}\|_{\ell_q^s(\mathcal{M}_p^u(\mathbb{R}^n))} \\ &\lesssim \left\{ \sum_{j \in \mathbb{Z}_+} 2^{jsq} \left\| \sup_{y \in \mathbb{R}^n} \frac{|\phi_j * g(y)|}{(1 + 2^j|\cdot - y|)^N} \right\|_{\mathcal{M}_p^u(\mathbb{R}^n)} \right\}^{1/q}, \end{aligned}$$

which, together with the characterization of $\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)$ via the Peetre maximal function (see, for example, [51, Subsection 11.2]) and the choice of g , further implies that

$$\|\mathcal{E}f\|_{\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)} \lesssim \|g\|_{\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{N}_{u,p,q}^s(\Omega)} + \varepsilon.$$

Letting $\varepsilon \rightarrow 0$, we then know that \mathcal{E} is a bounded linear operator from $\mathcal{N}_{u,p,q}^s(\Omega)$ into $\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)$.

Finally, since the supports of ψ_0 and ψ lie in $-K$, it follows that

$$\mathcal{E}f|_\Omega = \sum_{j \in \mathbb{Z}_+} \psi_j * \phi_j * f = f$$

in $D'(\Omega)$ (see page 249 of [74]). Thus, \mathcal{E} is the desired extension operator from $\mathcal{N}_{u,p,q}^s(\Omega)$ into $\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)$. This finishes the proof of Proposition 4.13. \square

Remark 4.14. Let $s \in \mathbb{R}$, $q \in (0, \infty]$, $0 < p \leq u < \infty$ and $\Omega \subset \mathbb{R}^n$ be an interval if $n = 1$ or a Lipschitz domain if $n \geq 2$. One advantage of the construction of \mathcal{E} in (4.39) lies in that, if $f \in \mathcal{N}_{u,p,q}^s(\Omega)$ and $g := \mathcal{E}f$ is the extension of f to \mathbb{R}^n , then $\partial^\alpha g = \mathcal{E}(\partial^\alpha f)$ for all $\alpha \in \mathbb{Z}_+^n$ (see [99, (4.70)]).

By the observation in the above remark, we have the following characterization of $\mathcal{N}_{u,p,q}^s(\Omega)$.

Proposition 4.15. Let $q \in (0, \infty]$, $0 < p \leq u < \infty$, $s = \sigma + k$ with $\sigma \in \mathbb{R}$ and $k \in \mathbb{N}$, and $\Omega \subset \mathbb{R}^n$ be an interval if $n = 1$ or a Lipschitz domain if $n \geq 2$. Then

$$\mathcal{N}_{u,p,q}^s(\Omega) = \{f \in \mathcal{N}_{u,p,q}^\sigma(\Omega) : \partial^\alpha f \in \mathcal{N}_{u,p,q}^\sigma(\Omega), |\alpha| \leq k\}$$

and there exists a positive constant $C \in [1, \infty)$ such that, for all $f \in \mathcal{N}_{u,p,q}^s(\Omega)$,

$$C^{-1} \|f\|_{\mathcal{N}_{u,p,q}^s(\Omega)} \leq \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{\mathcal{N}_{u,p,q}^\sigma(\Omega)} \leq C \|f\|_{\mathcal{N}_{u,p,q}^s(\Omega)}.$$

Proof. It is known from [92, Theorem 2.15(i)] that Proposition 4.15 holds true when $\Omega = \mathbb{R}^n$. From this and Remark 4.14, we deduce that, for all $f \in \mathcal{N}_{u,p,q}^s(\Omega)$,

$$\sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{\mathcal{N}_{u,p,q}^\sigma(\Omega)} \leq \sum_{|\alpha| \leq k} \|\partial^\alpha(\mathcal{E}f)\|_{\mathcal{N}_{u,p,q}^\sigma(\mathbb{R}^n)} \lesssim \|\mathcal{E}f\|_{\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{N}_{u,p,q}^s(\Omega)}.$$

Conversely, by [92, Theorem 2.15(i)] and Remark 4.14 again, we see that

$$\begin{aligned} \|f\|_{\mathcal{N}_{u,p,q}^s(\Omega)} &\leq \|\mathcal{E}f\|_{\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)} \lesssim \sum_{|\alpha| \leq k} \|\partial^\alpha(\mathcal{E}f)\|_{\mathcal{N}_{u,p,q}^\sigma(\mathbb{R}^n)} \sim \sum_{|\alpha| \leq k} \|\mathcal{E}(\partial^\alpha f)\|_{\mathcal{N}_{u,p,q}^\sigma(\mathbb{R}^n)} \\ &\lesssim \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{\mathcal{N}_{u,p,q}^\sigma(\Omega)}, \end{aligned}$$

which completes the proof of Proposition 4.15. \square

Now we are ready to prove Theorem 2.45.

Proof of Theorem 2.45. For brevity, we put

$$N_{u,p,q}^s(\Omega) := \left\{ f \in \mathcal{N}_{u,p,q}^s(\Omega) : D^\alpha f \in \mathcal{N}_{u,p,q}^s(\Omega) \text{ for all } \alpha \in \mathbb{Z}_+^n \right\}.$$

We claim that $N_{u,p,q}^s(\Omega)$ is independent of s , u , p and q . Indeed, by Proposition 4.15, we know that, if $f \in N_{u,p,q}^s(\Omega)$, then $f \in \mathcal{N}_{u,p,q}^\sigma(\Omega)$ for any $\sigma \in \mathbb{R}$, and hence $\mathcal{E}f \in \mathcal{N}_{u,p,q}^\sigma(\mathbb{R}^n)$ for any $\sigma \in \mathbb{R}$. In addition, we mention the embedding

$$\mathcal{N}_{u,p,q}^s(\mathbb{R}^n) \hookrightarrow C_{ub}(\mathbb{R}^n) \quad \text{if } s > n/p;$$

see Kozono and Yamazaki [41] or Sickel [86]. Combining these two arguments, we find that

$$N_{u,p,q}^s(\Omega) := \left\{ f \in C^\infty(\Omega) : D^\alpha f \in C(\Omega) \text{ for all } \alpha \in \mathbb{Z}_+^n \right\}$$

and this proves the above claim. Hence, we may write $N(\Omega) := N_{u,p,q}^s(\Omega)$. This implies that

$$\begin{aligned} \hat{\mathcal{N}}_{u,p,q}^s(\Omega) &= \overline{N(\Omega)}^{\mathcal{N}_{u,p,q}^s(\Omega)} = \overline{N_{u_0,p_0,q_0}^{s_0}(\Omega) \cap N_{u_1,p_1,q_1}^{s_1}(\Omega)}^{\mathcal{N}_{u,p,q}^s(\Omega)} \\ &\hookrightarrow \overline{\mathcal{N}_{u_0,p_0,q_0}^{s_0}(\Omega) \cap \mathcal{N}_{u_1,p_1,q_1}^{s_1}(\Omega)}^{\mathcal{N}_{u,p,q}^s(\Omega)}. \end{aligned}$$

On the other hand, by Lemma 2.26 and its proof, we know that

$$\hat{\mathcal{N}}_{u_i,p_i,q_i}^{s_i}(\Omega) = \mathcal{N}_{u_i,p_i,q_i}^{s_i}(\Omega)$$

and $S_N f \rightarrow f$ as $N \rightarrow \infty$ in $\mathcal{N}_{u_i,p_i,q_i}^{s_i}(\Omega)$, $i \in \{0, 1\}$, if $q_0, q_1 \in (0, \infty)$. Thus, any $f \in \mathcal{N}_{u_0,p_0,q_0}^{s_0}(\Omega) \cap \mathcal{N}_{u_1,p_1,q_1}^{s_1}(\Omega)$ can be approximated by $S_N f \in N(\Omega)$ in $\mathcal{N}_{u,p,q}^s(\Omega)$, which further implies that

$$\hat{\mathcal{N}}_{u,p,q}^s(\Omega) = \overline{\mathcal{N}_{u_0,p_0,q_0}^{s_0}(\Omega) \cap \mathcal{N}_{u_1,p_1,q_1}^{s_1}((0,1)^n)}^{\mathcal{N}_{u,p,q}^s(\Omega)}.$$

Now we continue with an application of Theorem 2.12(ii), which, together with the existence of a bounded linear extension operator

$$\mathcal{E} \in \mathcal{L}(\mathcal{N}_{u_i,p_i,q_i}^{s_i}(\Omega), \mathcal{N}_{u_i,p_i,q_i}^{s_i}(\mathbb{R}^n)), \quad i \in \{0, 1\},$$

in Proposition 4.13 (see also Sawano [77] for the case of smooth domains), and the method of the retraction and the coretraction, implies that

$$\langle \mathcal{N}_{u_0,p_0,q_0}^{s_0}(\Omega), \mathcal{N}_{u_1,p_1,q_1}^{s_1}(\Omega), \Theta \rangle = \mathcal{N}_{u,p,q}^s(\Omega),$$

if $p_0 u_1 = p_1 u_0$. Proposition 2.20 makes clear that

$$\overline{\mathcal{N}_{u_0,p_0,q_0}^{s_0}(\Omega) \cap \mathcal{N}_{u_1,p_1,q_1}^{s_1}(\Omega)}^{\mathcal{N}_{u,p,q}^s(\Omega)} = \langle \mathcal{N}_{u_0,p_0,q_0}^{s_0}(\Omega), \mathcal{N}_{u_1,p_1,q_1}^{s_1}(\Omega) \rangle_{\Theta}.$$

This finishes the proof of Theorem 2.45. □

4.4 Proofs of results in Subsection 2.4

4.4.1 Proofs of results in Subsection 2.4.1

For reader's convenience, we give proofs of Propositions 2.52, 2.53 and 2.54. Notice that, in our references [40] and [39], the additional assumption that $X_0 \cap X_1$ is dense in X_j , $j \in \{0, 1\}$, is used. In the proofs given below, we avoid this assumption.

Proof of Proposition 2.52. Let $\{f_n\}_{n \in \mathbb{N}}$ denote a Cauchy sequence in $\mathcal{A}(X_0, X_1)$. Since $X_0 + X_1$ is analytically convex, we conclude, from Proposition 2.47, that, for any $z \in S$,

$$\|f_n(z) - f_m(z)\|_{X_0+X_1} \lesssim \max_{t \in \mathbb{R}} \left\{ \|f_n(it) - f_m(it)\|_{X_0}, \|f_n(1+it) - f_m(1+it)\|_{X_1} \right\}.$$

Hence, for any $z \in S$, there exists a limit $f(z) = \lim_{n \rightarrow \infty} f_n(z) \in X_0 + X_1$, due to the completeness of $X_0 + X_1$. Because this convergence is uniform on any open set $U \subset S_0$, we conclude, by using Proposition 2.48, that f is an analytic function. On the other hand, since functions $f_n(it)$ and $f_n(1+it)$ are continuous and bounded on $t \in \mathbb{R}$ and the boundedness is uniform in $n \in \mathbb{N}$, their limit functions $f(it)$ and $f(1+it)$ are continuous and bounded on $t \in \mathbb{R}$ as well, i. e., $f \in \mathcal{A}(X_0, X_1)$. This proves (i).

Part (ii) is a consequence of the following observation. Let \mathcal{N}_Θ be the set of all functions $f \in \mathcal{A}(X_0, X_1)$ such that $f(\Theta) = 0$. Consequently, \mathcal{N}_Θ is a closed linear subspace of $\mathcal{A}(X_0, X_1)$. Since $[X_0, X_1]_\Theta$ is isomorphic to $\mathcal{A}(X_0, X_1)/\mathcal{N}_\Theta$, it is a complete space. This finishes the proof of (ii) and hence Proposition 2.52. \square

Proof of Proposition 2.53. Temporarily we assume $\|T\|_{X_j \rightarrow Y_j} > 0$, $j \in \{0, 1\}$. For $\Theta \in (0, 1)$, we define

$$g(z) := \left(\frac{\|T\|_{X_0 \rightarrow Y_0}}{\|T\|_{X_1 \rightarrow Y_1}} \right)^{z-\Theta} T f(z), \quad z \in S, \quad f \in \mathcal{A}(X_0, X_1).$$

Hence, for all $t \in \mathbb{R}$,

$$\|g(it)\|_{Y_0} \leq \left(\frac{\|T\|_{X_0 \rightarrow Y_0}}{\|T\|_{X_1 \rightarrow Y_1}} \right)^{-\Theta} \|T\|_{X_0 \rightarrow Y_0} \|f(it)\|_{X_0}$$

and, similarly,

$$\|g(1+it)\|_{Y_1} \leq \left(\frac{\|T\|_{X_0 \rightarrow Y_0}}{\|T\|_{X_1 \rightarrow Y_1}} \right)^{1-\Theta} \|T\|_{X_1 \rightarrow Y_1} \|f(1+it)\|_{X_1}.$$

This implies that g belongs to $\mathcal{A}(Y_0, Y_1)$. Let $x := f(\Theta) \in [X_0, X_1]_\Theta$. Then

$$y := g(\Theta) = T f(\Theta) \in [Y_0, Y_1]_\Theta$$

and

$$\|y\|_{[Y_0, Y_1]_\Theta} \leq \|g\|_{\mathcal{A}(Y_0, Y_1)} \leq \|T\|_{X_0 \rightarrow Y_0}^{1-\Theta} \|T\|_{X_1 \rightarrow Y_1}^\Theta \|f\|_{\mathcal{A}(X_0, X_1)}.$$

Taking the infimum in both sides of the above inequality with respect to all $f \in \mathcal{A}(X_0, X_1)$ such that $f(\Theta) = x$, we obtain the desired conclusion. If $\|T\|_{X_0 \rightarrow Y_0} = 0$ or $\|T\|_{X_1 \rightarrow Y_1} = 0$, then one has to replace this quantity by $\varepsilon \in (0, \infty)$ in the definition of g and finally, consider $\varepsilon \downarrow 0$, the details being omitted. This finishes the proof of Proposition 2.52. \square

Proof of Proposition 2.54. One may use Triebel's arguments in the proof of [94, Theorem 1.2.4], since the closed graph theorem remains true in the context of quasi-Banach spaces, the details being omitted. This finishes the proof of Proposition 2.54. \square

4.4.2 Proofs of results in Subsection 2.4.3

Proof of Theorem 2.68

By Remark 2.92(i), we know that all spaces under consideration are lattice r -convex for some r . Proposition 2.64 and Theorem 2.5(i) yield

$$[\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)]_\Theta^i \hookrightarrow [\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n)]^{1-\Theta} [\mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)]^\Theta \hookrightarrow \mathcal{M}_p^u(\mathbb{R}^n).$$

This shows (i) of Theorem 2.68.

To prove (ii), first observe that Theorem 2.5(iii) implies that this embedding is proper if $u_0 p_1 \neq u_1 p_0$. In case $u_0 p_1 = u_1 p_0$, we derive, from Corollaries 4.4 and 2.65, that

$$[\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)]_\Theta^i = \langle \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) \rangle_\Theta.$$

Now the desired conclusion (ii) of Theorem 2.68 follows from Corollary 2.38 and Theorem 2.40, the details being omitted. This finishes the proof of (ii) and hence Theorem 2.68.

Remark 4.16. Theorem 2.68 shows that [46, Theorem 3(ii)] is not correct. However, let us mention that [46] has been our main source for the cases $u_0 p_1 \neq u_1 p_0$.

Proofs of Propositions 2.69 and 2.71

In both conclusions of Proposition 2.69, the first embedding is a consequence of the definition of the complex method and of the inner complex method. The second embedding has been proved in [111]. We give a sketch for the convenience of the reader. In the proofs of [111, Propositions 2.6 and 2.7], the condition $\tau_0 p_0 = \tau_1 p_1$ is not used to establish the embeddings

$$[a_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n)]^{1-\Theta} [a_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)]^\Theta \hookrightarrow a_{p, q}^{s, \tau}(\mathbb{R}^n), \quad a \in \{f, b\};$$

see also Proposition 2.7. This has to be combined with [111, Proposition 1.10]:

$$[a_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n), a_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)]_\Theta \hookrightarrow [a_{p_0, q_0}^{s_0, \tau_0}(\mathbb{R}^n)]^{1-\Theta} [a_{p_1, q_1}^{s_1, \tau_1}(\mathbb{R}^n)]^\Theta \hookrightarrow a_{p, q}^{s, \tau}(\mathbb{R}^n).$$

By these and Proposition 5.8, together with an argument similar to that used in the proof of Theorem 2.12, we then obtain the second embeddings of Proposition 2.69, the details being omitted. Hence, Proposition 2.69 is proved.

Concerning the proof of Proposition 2.71, we argue in the same way as the proof of Proposition 2.69. In case of the embedding

$$[n_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n)]^{1-\Theta} [n_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n)]^\Theta \hookrightarrow n_{u, p, q}^s(\mathbb{R}^n),$$

the restriction $p_0 u_1 = u_0 p_1$ was not used in [111, Proposition 2.8]; see Proposition 2.8. Furthermore, the embedding

$$[n_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n), n_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n)]_\Theta \hookrightarrow [n_{u_0, p_0, q_0}^{s_0}(\mathbb{R}^n)]^{1-\Theta} [n_{u_1, p_1, q_1}^{s_1}(\mathbb{R}^n)]^\Theta$$

has been proved in [111, Proposition 1.10]. The proof of Proposition 2.71 is then finished.

Proof of Theorem 2.72

Because of

$$\langle \mathcal{M}_{p_0}^{u_0}((0, 1)^n), \mathcal{M}_{p_1}^{u_1}((0, 1)^n), \Theta \rangle = \mathcal{M}_p^u((0, 1)^n)$$

(see (4.38)) and

$$[\mathcal{M}_{p_0}^{u_0}((0, 1)^n)]^{1-\Theta} [\mathcal{M}_{p_1}^{u_1}((0, 1)^n)]^\Theta = \mathcal{M}_p^u((0, 1)^n)$$

(see [56, (2.3)]), it follows, from Corollary 2.65, that

$$[\mathcal{M}_{p_0}^{u_0}((0, 1)^n), \mathcal{M}_{p_1}^{u_1}((0, 1)^n)]_\Theta^i = \langle \mathcal{M}_{p_0}^{u_0}((0, 1)^n), \mathcal{M}_{p_1}^{u_1}((0, 1)^n) \rangle_\Theta.$$

Now we apply Theorem 2.44 to obtain the desired conclusion, which completes the proof of Theorem 2.72.

4.5 Proofs of results in Subsection 2.6

Proof of Corollary 2.85

The conditions in Corollary 2.85 are guaranteeing that

$$A_{p_i, q_i}^{s_i, \tau_i}(\mathbb{R}^n) = B_{\infty, \infty}^{s_i + n(\tau_i - 1/p_i)}(\mathbb{R}^n), \quad A \in \{B, F\}, \quad i \in \{0, 1\};$$

see Proposition 5.5(iii). Now it suffices to recall

$$(B_{\infty, \infty}^{s_0 + n(\tau_0 - 1/p_0)}(\mathbb{R}^n), B_{\infty, \infty}^{s_1 + n(\tau_1 - 1/p_1)}(\mathbb{R}^n))_{\Theta, q} = B_{\infty, q}^{s + n(\tau - 1/p)}(\mathbb{R}^n);$$

see [96, Theorem 2.4.2]. This finishes the proof of Corollary 2.85.

Proof of Lemma 2.87

Part (i) is proved in Lemma 2.91 and Remark 2.92(i). For (ii), we refer the reader to Lemarié-Rieussiet [46]. Concerning (iii), we use

$$L_p(\mathbb{R}^n) = (L_{p_0}(\mathbb{R}^n), L_{p_1}(\mathbb{R}^n))_{\Theta, \infty}$$

because of $p_0 = p_1$. Hence, taking Proposition 2.79(ii) into account, we may employ Lemma 2.91 with this functor $(\cdot, \cdot)_{\Theta, \infty}$. Choosing $T = I$, we obtain the desired conclusion in (iii), which completes the proof of Lemma 2.87.

Proof of Theorem 2.88

In the case (a), we see that $p_0 = p_1 = p$ and $u_0 = u_1 = u$, and hence

$$(\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n))_{\Theta, q} = (\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n))_{\Theta, q} = \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) = \mathcal{M}_p^u(\mathbb{R}^n).$$

In the case (b), we have

$$(\mathcal{M}_{p_0}^{p_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{p_1}(\mathbb{R}^n))_{\Theta, q} = (L_{p_0}(\mathbb{R}^n), L_{p_1}(\mathbb{R}^n))_{\Theta, p} = L_p(\mathbb{R}^n) = \mathcal{M}_p^p(\mathbb{R}^n).$$

Next we argue by contradiction to show, if neither (a) nor (b) is true, then

$$(\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n))_{\Theta, q} \neq \mathcal{M}_p^u(\mathbb{R}^n).$$

Let us assume $(\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n))_{\Theta, q} = \mathcal{M}_p^u(\mathbb{R}^n)$ for some $\Theta \in (0, 1)$ and some $q \in (0, \infty]$. We now consider two cases.

Step 1. Let $q \in (0, \infty)$. Then [7, Theorem 3.4.2] yields that $\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) \cap \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n)$ must be dense in $\mathcal{M}_p^u(\mathbb{R}^n)$. Now, applying Lemma 2.37, we see that the above assumption yields a contradiction.

Step 2. Let $q = \infty$. Lemarié-Rieussiet [46] has proved that

$$\mathcal{M}_p^u(\mathbb{R}^n) \hookrightarrow (\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n))_{\Theta, \infty} \iff p_0 u_1 = p_1 u_0.$$

Also in the quoted article [46], one can find that

$$(\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n), \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n))_{\Theta, \infty} \hookrightarrow \mathcal{M}_p^u(\mathbb{R}^n)$$

implies $p_0 = p_1$. Hence, our assumption yields $p_0 = p_1$ and $u_0 = u_1$, i. e.,

$$\mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n) = L_{p_0}(\mathbb{R}^n) \quad \text{and} \quad \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n) = L_{p_0}(\mathbb{R}^n).$$

Of course, $(L_{p_0}(\mathbb{R}^n), L_{p_0}(\mathbb{R}^n))_{\Theta, \infty} = L_{p_0}(\mathbb{R}^n)$, but this case is already excluded in Step 2. This finishes the proof of Theorem 2.88.

Proof of Lemma 2.89

The key tool is [94, Theorem 1.4.2], which holds true also for quasi-Banach cases; see [94, Remark 1.4.2/3]. Let $\eta := \Theta p/p_1$. Then $p = (1 - \eta)p_0 + \eta p_1$ follows. Applying [94, Theorem 1.4.2] with $q = 1$, we obtain, for any sequence $a := \{a_j\}_{j \in \mathbb{Z}_+}$, $a_j \in X_0 + X_1$,

$$\begin{aligned} \|a\|_{(\ell_{p_0}^{s_0}(X_0), \ell_{p_1}^{s_1}(X_1))_{\Theta, p}}^p &\asymp \int_0^\infty t^{-\eta} \inf_{a = a_j^0 + a_j^1} \sum_{j \in \mathbb{Z}_+} [2^{js_0 p_0} \|a_j^0\|_{X_0}^{p_0} + t 2^{js_1 p_1} \|a_j^1\|_{X_1}^{p_1}] \frac{dt}{t} \\ &\asymp \sum_{j \in \mathbb{Z}_+} \int_0^\infty t^{-\eta} \inf_{a_j = a_j^0 + a_j^1} [2^{js_0 p_0} \|a_j^0\|_{X_0}^{p_0} + t 2^{js_1 p_1} \|a_j^1\|_{X_1}^{p_1}] \frac{dt}{t}. \end{aligned}$$

By a change of variable $y := (k_j)^{-1}t$ with $k_j := 2^{j(s_0 p_0 - s_1 p_1)}$, we conclude that

$$\|a\|_{(\ell_{p_0}^{s_0}(X_0), \ell_{p_1}^{s_1}(X_1))_{\Theta, p}}^p \asymp \sum_{j \in \mathbb{Z}_+} \int_0^\infty y^{-\eta} \inf_{a_j = a_j^0 + a_j^1} [k_j^{-\eta} 2^{js_0 p_0} \|a_j^0\|_{X_0}^{p_0} + y k_j^{1-\eta} 2^{js_1 p_1} \|a_j^1\|_{X_1}^{p_1}] \frac{dy}{y}.$$

Observe that s and p satisfy the following identities

$$\eta(s_1 p_1 - s_0 p_0) + s_0 p_0 = \Theta p s_1 + (1 - \Theta p/p_1) s_0 p_0 = [\Theta s_1 + (1 - \Theta) s_0] p = s p$$

and

$$(1 - \eta)(s_0 p_0 - s_1 p_1) + s_1 p_1 = s p.$$

Hence

$$\begin{aligned} \|a\|_{(\ell_{p_0}^{s_0}(X_0), \ell_{p_1}^{s_1}(X_1))_{\Theta, p}}^p &\asymp \sum_{j \in \mathbb{Z}_+} 2^{j s p} \int_0^\infty y^{-\eta} \inf_{a_j = a_j^0 + a_j^1} [\|a_j^0\|_{X_0}^{p_0} + y \|a_j^1\|_{X_1}^{p_1}] \frac{dy}{y} \\ &\asymp \|a\|_{\ell_p^s((X_0, X_1)_{\Theta, p})}^p, \end{aligned}$$

which proves the desired conclusion and hence completes the proof of Lemma 2.89.

4.6 Proof of Lemma 2.91

Let B denote any ball in \mathbb{R}^n . Since $T \in \mathcal{L}(X_0, \mathcal{M}_{p_0}^{u_0}(\mathbb{R}^n)) \cap \mathcal{L}(X_1, \mathcal{M}_{p_1}^{u_1}(\mathbb{R}^n))$, we know that

$$\chi_B T \in \mathcal{L}(X_0, L_{p_0}(B)) \cap \mathcal{L}(X_1, L_{p_1}(B))$$

and

$$\|\chi_B T\|_{X_i \rightarrow L_{p_i}(B)} \leq |B|^{\frac{1}{p_i} - \frac{1}{u_i}} M_i, \quad i \in \{0, 1\}.$$

Moreover, by (2.25), we see that

$$\chi_B T \in \mathcal{L}[F(X_0, X_1), F(L_{p_0}(B), L_{p_1}(B))] \hookrightarrow \mathcal{L}[F(X_0, X_1), L_p(B)]$$

and

$$\|\chi_B T\|_{F(X_0, X_1) \rightarrow L_p(B)} \leq C_F \left[|B|^{\frac{1}{p_0} - \frac{1}{u_0}} M_0 \right]^{1-\Theta} \left[|B|^{\frac{1}{p_1} - \frac{1}{u_1}} M_1 \right]^\Theta$$

with a positive constant C_F independent of B . Let $f \in F(X_0, X_1)$. Then we conclude $Tf \in L_p(B)$ and

$$\left[\int_B |Tf(x)|^p dx \right]^{1/p} \leq C_F M_0^{1-\Theta} M_1^\Theta |B|^{\frac{1}{p} - \frac{1}{u}} \|f\|_{F(X_0, X_1)}.$$

This finishes the proof of Lemma 2.91.

4.7 Proof of Theorem 3.3

Similar to the arguments used in the proof of Theorem 2.12, we first consider the sequence spaces related to $A_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$.

Definition 4.17. Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$ and $p, q \in (0, \infty]$.

The sequence space $b_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$ is defined as the collection of all complex-valued sequences $t := \{t_Q\}_{Q \in \mathcal{Q}^*}$ such that

$$\|t\|_{b_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)} := \sup_{P \in \mathcal{Q}^*} \frac{1}{|P|^\tau} \left[\sum_{j=j_P}^\infty 2^{j(s+\frac{n}{2})q} \left\{ \int_P \left[\sum_{\ell(Q)=2^{-j}} |t_Q| \chi_Q(x) \right]^p dx \right\}^{\frac{q}{p}} \right]^{\frac{1}{q}} < \infty$$

with the usual modifications made in case $p = \infty$ and/or $q = \infty$.

The sequence space $f_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$ with $p \in (0, \infty)$ is defined as the collection of all complex-valued sequences $t := \{t_Q\}_{Q \in \mathcal{Q}^*}$ such that

$$\|t\|_{f_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)} := \sup_{P \in \mathcal{Q}^*} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=j_P}^\infty 2^{j(s+\frac{n}{2})q} \sum_{\ell(Q)=2^{-j}} |t_Q|^q \chi_Q(x) \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}} < \infty$$

with the usual modification made when $q = \infty$.

As before, we use $a_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$ to denote either $b_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$ or $f_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$. Notice that the space $a_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$ coincides with the sequence space $\mathcal{L}^{n(\tau-1/p)} a_{p,q}^{s+n/2}(\mathbb{R}^n)$ related to $\mathcal{L}^{n(\tau-1/p)} A_{p,q}^s(\mathbb{R}^n)$; see [100, Definition 1.30].

It is easy to see that the space $a_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$ has the following equivalent characterization, the details being omitted.

Proposition 4.18. *Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$ and $p, q \in (0, \infty]$. A sequence $t := \{t_Q\}_{Q \in \mathcal{Q}^*} \in a_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$ if and only if*

$$\|t\|_{a_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)} := \sup_{\ell \in \mathbb{Z}^n} \left\| \{t_Q \chi_\ell(Q)\}_{Q \in \mathcal{Q}^*} \right\|_{a_{p,q}^{s,\tau}(\mathbb{R}^n)} < \infty,$$

where $\chi_\ell := \chi_{\{R \in \mathcal{Q}^*: R \subset Q_{0,\ell}\}}$. Moreover, $\|\cdot\|_{a_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)}$ is equivalent to $\|\cdot\|_{a_{p,q}^{s,\tau}(\mathbb{R}^n)}$.

Applying Proposition 4.18, one can show that the Calderón product property of $a_{p,q}^{s,\tau}(\mathbb{R}^n)$ in Proposition 2.7 is also true for $a_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$.

Proposition 4.19. *Let all parameters be as in Proposition 2.7. Then*

$$\left[a_{p_0,q_0,\text{unif}}^{s_0,\tau_0}(\mathbb{R}^n) \right]^{1-\Theta} \left[a_{p_1,q_1,\text{unif}}^{s_1,\tau_1}(\mathbb{R}^n) \right]^\Theta = a_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$$

Proof. The embedding

$$\left[a_{p_0,q_0,\text{unif}}^{s_0,\tau_0}(\mathbb{R}^n) \right]^{1-\Theta} \left[a_{p_1,q_1,\text{unif}}^{s_1,\tau_1}(\mathbb{R}^n) \right]^\Theta \hookrightarrow a_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$$

follows directly from the Hölder inequality.

Now we prove the inverse embedding. Let $t := \{t_Q\}_{Q \in \mathcal{Q}^*} \in a_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)$. By Proposition 4.18, we know that, for all $\ell \in \mathbb{Z}^n$, $\{t_Q \chi_\ell(Q)\}_{Q \in \mathcal{Q}^*} \in a_{p,q}^{s,\tau}(\mathbb{R}^n)$ and

$$\left\| \{t_Q \chi_\ell(Q)\}_{Q \in \mathcal{Q}^*} \right\|_{a_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim \|t\|_{a_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)}.$$

Then, by Proposition 2.7, we conclude that there exist $t^{0,\ell} := \{t_Q^{0,\ell}\}_{Q \in \mathcal{Q}^*} \in a_{p_0,q_0}^{s_0,\tau_0}(\mathbb{R}^n)$ and $t^{1,\ell} := \{t_Q^{1,\ell}\}_{Q \in \mathcal{Q}^*} \in a_{p_1,q_1}^{s_1,\tau_1}(\mathbb{R}^n)$ such that $|t_Q \chi_\ell(Q)| \leq |t_Q^{0,\ell}|^{1-\Theta} |t_Q^{1,\ell}|^\Theta$ for all $Q \in \mathcal{Q}^*$ and

$$\|t^{0,\ell}\|_{a_{p_0,q_0}^{s_0,\tau_0}(\mathbb{R}^n)}^{1-\Theta} \|t^{1,\ell}\|_{a_{p_1,q_1}^{s_1,\tau_1}(\mathbb{R}^n)}^\Theta \lesssim \left\| \{t_Q \chi_\ell(Q)\}_{Q \in \mathcal{Q}^*} \right\|_{a_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim \|t\|_{a_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)}.$$

Define t^0 and t^1 by setting, for all $Q \in \mathcal{Q}^*$, $t_Q^0 := \sum_{\ell \in \mathbb{Z}^n} |t_Q^{0,\ell}| \chi_\ell(Q)$ and $t_Q^1 := \sum_{\ell \in \mathbb{Z}^n} |t_Q^{1,\ell}| \chi_\ell(Q)$. Then, by the Hölder inequality, we see that

$$|t_Q| = \sum_{\ell \in \mathbb{Z}^n} |t_Q \chi_\ell(Q)| \leq \sum_{\ell \in \mathbb{Z}^n} |t_Q^{0,\ell}|^{1-\Theta} |t_Q^{1,\ell}|^\Theta \chi_\ell(Q) \leq |t_Q^0|^{1-\Theta} |t_Q^1|^\Theta.$$

Moreover, we have

$$\begin{aligned} \|t^i\|_{a_{p_i,q_i}^{s_i,\tau_i}(\mathbb{R}^n)} &= \sup_{m \in \mathbb{Z}^n} \left\| \{t_Q^i \chi_m(Q)\}_{Q \in \mathcal{Q}^*} \right\|_{a_{p_i,q_i}^{s_i,\tau_i}(\mathbb{R}^n)} \\ &= \sup_{m \in \mathbb{Z}^n} \|t^{i,m}\|_{a_{p_i,q_i}^{s_i,\tau_i}(\mathbb{R}^n)} \lesssim \|t\|_{a_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n)}, \end{aligned}$$

which implies that $a_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n) \hookrightarrow [a_{p_0,q_0}^{s_0,\tau_0}(\mathbb{R}^n)]^{1-\Theta} [a_{p_1,q_1}^{s_1,\tau_1}(\mathbb{R}^n)]^\Theta$, and hence completes the proof of Proposition 4.19. \square

Repeating the arguments used in the proofs of Theorem 4.4 and Corollary 4.7, with Proposition 2.7 replaced by Proposition 4.19, we obtain the following interpolation formulas, the details being omitted.

Theorem 4.20. *Let $\Theta \in (0, 1)$, $s_i \in \mathbb{R}$, $\tau_i \in [0, \infty)$, $p_i, q_i \in (0, \infty]$, $i \in \{1, 2\}$, such that*

$$s = (1 - \Theta)s_0 + \Theta s_1, \quad \tau = (1 - \Theta)\tau_0 + \Theta \tau_1, \quad \frac{1}{p} = \frac{1 - \Theta}{p_0} + \frac{\Theta}{p_1}, \quad \frac{1}{q} = \frac{1 - \Theta}{q_0} + \frac{\Theta}{q_1} \quad \text{and} \quad \frac{\tau_0}{p_1} = \frac{\tau_1}{p_0}.$$

Then

$$\left\langle a_{p_0, q_0, \text{unif}}^{s_0, \tau_0}(\mathbb{R}^n), a_{p_1, q_1, \text{unif}}^{s_1, \tau_1}(\mathbb{R}^n) \right\rangle_{\Theta} = (a_{p, q, \text{unif}}^{s, \tau}(\mathbb{R}^n))^{\#}$$

and

$$\left\langle a_{p_0, q_0, \text{unif}}^{s_0, \tau_0}(\mathbb{R}^n), a_{p_1, q_1, \text{unif}}^{s_1, \tau_1}(\mathbb{R}^n), \Theta \right\rangle = a_{p, q, \text{unif}}^{s, \tau}(\mathbb{R}^n).$$

Assume further that $p_i, q_i \in [1, \infty]$. Then

$$\langle a_{p_0, q_0, \text{unif}}^{s_0, \tau_0}(\mathbb{R}^n), a_{p_1, q_1, \text{unif}}^{s_1, \tau_1}(\mathbb{R}^n) \rangle_{\Theta} = (a_{p, q, \text{unif}}^{s, \tau}(\mathbb{R}^n))^{\#} = [a_{p_0, q_0, \text{unif}}^{s_0, \tau_0}(\mathbb{R}^n), a_{p_1, q_1, \text{unif}}^{s_1, \tau_1}(\mathbb{R}^n)]_{\Theta}.$$

Theorem 3.3 is then an immediate consequence of Theorem 4.20 and the wavelet characterization of the spaces $A_{p, q, \text{unif}}^{s, \tau}(\mathbb{R}^n) = \mathcal{L}^{n(\tau-1/p)} A_{p, q}^s(\mathbb{R}^n)$ in [100, Theorem 1.32], the details being omitted.

5 Appendix – Function spaces

For the convenience of the reader, we recall definitions and collect some properties of the function spaces considered in this article.

5.1 Besov-type and Triebel-Lizorkin-type spaces

Besov-type and Triebel-Lizorkin-type spaces are generalizations of Besov and Triebel-Lizorkin spaces. As Besov and Triebel-Lizorkin spaces, also these more general scales of spaces can be introduced in very different ways. Here we use the Fourier-analytical approach. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be a radial function such that $\psi(x) \leq 1$ for all x ,

$$\psi(x) := 1 \quad \text{if } |x| \leq 1 \quad \text{and} \quad \psi(x) := 0 \quad \text{if } |x| \geq \frac{3}{2}. \quad (5.1)$$

Then, with $\varphi_0 := \psi$,

$$\varphi(x) := \varphi_0(x/2) - \varphi_0(x) \quad \text{and} \quad \varphi_j(x) := \varphi(2^{-j+1}x), \quad x \in \mathbb{R}^n, \quad j \in \mathbb{N}, \quad (5.2)$$

we have

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \quad \text{for all } x \in \mathbb{R}^n.$$

In what follows, for $f \in \mathcal{S}'(\mathbb{R}^n)$, we use $\mathcal{F}f$ to denote its *Fourier transform*, and $\mathcal{F}^{-1}f$ its *inverse Fourier transform*.

Definition 5.1. Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$, and $q \in (0, \infty]$.

(i) Let $p \in (0, \infty]$. The *Besov-type space* $B_{p, q}^{s, \tau}(\mathbb{R}^n)$ is defined as the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B_{p, q}^{s, \tau}(\mathbb{R}^n)} := \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{j=\max\{j_P, 0\}}^{\infty} 2^{jsq} \left[\int_P |\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)(x)|^p dx \right]^{q/p} \right\}^{1/q} < \infty$$

with the usual modifications made in case $p = \infty$ and/or $q = \infty$.

(ii) Let $p \in (0, \infty)$. The *Triebel-Lizorkin-type space* $F_{p, q}^{s, \tau}(\mathbb{R}^n)$ is defined as the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F_{p, q}^{s, \tau}(\mathbb{R}^n)} := \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=\max\{j_P, 0\}}^{\infty} 2^{jsq} |\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)(x)|^q \right]^{p/q} dx \right\}^{1/p} < \infty$$

with the usual modification made when $q = \infty$.

The above definition represents a natural extension of the Fourier-analytical approach to Besov and Triebel-Lizorkin spaces; see, e.g., [26], [68] and [96, 97]. The homogenous version of these spaces were introduced in [107, 108] in order to clarify the relation between Besov and Triebel-Lizorkin spaces and Q spaces (see [20, 25, 104, 105]).

Several classical spaces can be identified within these scales.

Proposition 5.2. Let $s \in \mathbb{R}$, $q \in (0, \infty]$ and $\tau \in [0, \infty)$.

(i) It holds true that

$$F_{p,q}^{s,0}(\mathbb{R}^n) = F_{p,q}^s(\mathbb{R}^n) \text{ (for } p \in (0, \infty)) \quad \text{and} \quad B_{p,q}^{s,0}(\mathbb{R}^n) = B_{p,q}^s(\mathbb{R}^n) \text{ (for } p \in (0, \infty)).$$

(ii) For all $p \in (0, \infty)$, $F_{p,q}^{s,1/p}(\mathbb{R}^n) = F_{\infty,q}^s(\mathbb{R}^n)$ ([26, Corollary 6.9]).

(iii) Let $p \in (0, \infty)$. If either $q \in (0, \infty)$ and $\tau \in (1/p, \infty)$ or $q = \infty$ and $\tau \in [1/p, \infty)$, then

$$A_{p,q}^{s,\tau}(\mathbb{R}^n) = B_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n), \quad A \in \{B, F\}.$$

(iv) For all $1 < p \leq u < \infty$ and $m \in \mathbb{N}$,

$$\mathcal{M}_p^u(\mathbb{R}^n) = F_{p,2}^{0,1/p-1/u}(\mathbb{R}^n) \quad \text{and} \quad W^m \mathcal{M}_p^u(\mathbb{R}^n) = F_{p,2}^{m,1/p-1/u}(\mathbb{R}^n),$$

where $W^m \mathcal{M}_p^u(\mathbb{R}^n)$ denotes the Morrey-Sobolev space of order m .

Remark 5.3. Proposition 5.2(i) is obvious, Proposition 5.2(ii) is a well-known result of Frazier and Jawerth [26, Corollary 6.9]. The identity in Proposition 5.2(iii) was recently proved in [109]. Finally, the Littlewood-Paley assertion in Proposition 5.2(iv) can be found in Mazzucato [59] and Sawano [76].

5.2 Besov-Morrey and Triebel-Lizorkin-Morrey spaces

The following spaces were first introduced and investigated in [41, 60, 78, 92].

Definition 5.4. Let $s \in \mathbb{R}$ and $q \in (0, \infty]$. Let $\{\varphi_j\}_{j \in \mathbb{Z}_+}$ be the smooth dyadic decomposition of unity as defined in as in (5.1) and (5.2).

(i) Let $0 < p \leq u \leq \infty$. The *Besov-Morrey space* $\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)$ is defined as the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)} := \left\{ \sum_{j \in \mathbb{Z}_+} 2^{jsq} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)\|_{\mathcal{M}_p^u(\mathbb{R}^n)}^q \right\}^{1/q} < \infty.$$

(ii) Let $0 < p \leq u < \infty$. The *Triebel-Lizorkin-Morrey space* $\mathcal{E}_{u,p,q}^s(\mathbb{R}^n)$ is defined as the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{E}_{u,p,q}^s(\mathbb{R}^n)} := \left\| \left[\sum_{j \in \mathbb{Z}_+} 2^{jsq} |\mathcal{F}^{-1}(\varphi_j \mathcal{F}f)|^q \right]^{1/q} \right\|_{\mathcal{M}_p^u(\mathbb{R}^n)} < \infty.$$

The following relations can be found in [115] and [81].

Proposition 5.5. Let $s \in \mathbb{R}$, $q \in (0, \infty]$ and $0 < p \leq u \leq \infty$. Then

$$\mathcal{E}_{u,p,q}^s(\mathbb{R}^n) = F_{p,q}^{s,1/p-1/u}(\mathbb{R}^n) \quad \text{if} \quad u < \infty$$

and

$$\mathcal{N}_{u,p,\infty}^s(\mathbb{R}^n) = B_{p,\infty}^{s,1/p-1/u}(\mathbb{R}^n).$$

In addition, it holds true that

$$\mathcal{N}_{u,p,q}^s(\mathbb{R}^n) \subsetneq B_{p,q}^{s,1/p-1/u}(\mathbb{R}^n) \quad \text{if} \quad q < \infty \quad \text{and} \quad u \neq p.$$

5.3 The local spaces of Triebel

We do not recall the original definition of the spaces $\mathcal{L}^r A_{p,q}^s(\mathbb{R}^n)$ given in Triebel [100, 1.3.1]. We simply state the following identity; see [116].

Proposition 5.6. *Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$ and $q \in (0, \infty]$. Let $p \in (0, \infty]$ if $A = B$ and $p \in (0, \infty)$ if $A = F$. Then*

$$A_{p,q,\text{unif}}^{s,\tau}(\mathbb{R}^n) = \mathcal{L}^{n(\tau-1/p)} A_{p,q}^s(\mathbb{R}^n)$$

in the sense of equivalent quasi-norms.

5.4 Associated sequence spaces

As in case of Besov-Triebel-Lizorkin spaces, discretizations play an important role. Either by the φ -transform or by the wavelet transform, one can relate these function spaces to sequence spaces. We recall their definitions.

Definition 5.7. Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$ and $q \in (0, \infty]$.

The sequence space $b_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $p \in (0, \infty]$ is defined as the collection of all sequences $t := \{t_Q\}_{Q \in \mathcal{Q}^*} \subset \mathbb{C}$ such that

$$\|t\|_{b_{p,q}^{s,\tau}(\mathbb{R}^n)} := \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left[\sum_{j=\max\{j_P, 0\}}^{\infty} 2^{j(s+\frac{n}{2})q} \left\{ \int_P \left[\sum_{\ell(Q)=2^{-j}} |t_Q| \chi_Q(x) \right]^p dx \right\}^{\frac{q}{p}} \right]^{\frac{1}{q}} < \infty$$

with the usual modifications made in case $p = \infty$ and/or $q = \infty$.

The sequence space $f_{p,q}^{s,\tau}(\mathbb{R}^n)$ with $p \in (0, \infty)$ is defined as the collection of all sequences $t := \{t_Q\}_{Q \in \mathcal{Q}^*} \subset \mathbb{C}$ such that

$$\|t\|_{f_{p,q}^{s,\tau}(\mathbb{R}^n)} := \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=\max\{j_P, 0\}}^{\infty} 2^{j(s+\frac{n}{2})q} \sum_{\ell(Q)=2^{-j}} |t_Q|^q \chi_Q(x) \right]^{\frac{p}{q}} dx \right\}^{\frac{1}{p}} < \infty$$

with the usual modification made when $q = \infty$.

To explain the connection between sequence spaces and function spaces, we employ wavelet decompositions. Wavelet bases in function spaces are a well-developed concept. We refer the reader to the monographs of Meyer [62], Wojtaszczyk [103] and Triebel [98, 99] for the general n -dimensional case (for the one-dimensional case we refer the reader to the books of Hernandez and Weiss [33] and Kahane and Lemarié-Rieuseut [36]). Let $\tilde{\phi}$ be an *orthonormal scaling function* on \mathbb{R} with compact support and of sufficiently high regularity. Let $\tilde{\psi}$ be one *corresponding orthonormal wavelet*. Then the *tensor product ansatz* yields a scaling function ϕ and associated wavelets $\psi_1, \dots, \psi_{2^n-1}$, all defined now on \mathbb{R}^n ; see, e. g., [103, Proposition 5.2]. We suppose

$$\phi \in C^{N_1}(\mathbb{R}^n) \quad \text{and} \quad \text{supp } \phi \subset [-N_2, N_2]^n \quad (5.3)$$

for some natural numbers N_1 and N_2 . This implies that

$$\psi_i \in C^{N_1}(\mathbb{R}^n) \quad \text{and} \quad \text{supp } \psi_i \subset [-N_3, N_3]^n, \quad i \in \{1, \dots, 2^n - 1\}$$

for some $N_3 \in \mathbb{N}$. For $k \in \mathbb{Z}^n$, $j \in \mathbb{Z}_+$ and $i \in \{1, \dots, 2^n - 1\}$, we shall use the standard abbreviations in this context:

$$\phi_{j,k}(x) := 2^{jn/2} \phi(2^j x - k) \quad \text{and} \quad \psi_{i,j,k}(x) := 2^{jn/2} \psi_i(2^j x - k), \quad x \in \mathbb{R}^n.$$

Furthermore, it is well known that

$$\int_{\mathbb{R}^n} \psi_{i,j,k}(x) x^\gamma dx = 0 \quad \text{if} \quad |\gamma| \leq N_1$$

(see [103, Proposition 3.1]) and

$$\{\phi_{0,k} : k \in \mathbb{Z}^n\} \cup \{\psi_{i,j,k} : k \in \mathbb{Z}^n, j \in \mathbb{Z}_+, i \in \{1, \dots, 2^n - 1\}\}$$

yields an *orthonormal basis* of $L_2(\mathbb{R}^n)$; see [62, Section 3.9] or [98, Section 3.1]. Namely, each function $f \in L_2(\mathbb{R}^n)$ admits a representation

$$f = \sum_{k \in \mathbb{Z}^n} \lambda_k \phi_{0,k} + \sum_{i=1}^{2^n-1} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \lambda_{i,j,k} \psi_{i,j,k}, \quad (5.4)$$

where $\lambda_k := \langle f, \phi_{0,k} \rangle$ and $\lambda_{i,j,k} := \langle f, \psi_{i,j,k} \rangle$. Concerning the mapping

$$\Phi : f \mapsto \{\lambda_k\}_k \cup \{\lambda_{i,j,k}\}_{i,j,k}$$

the following is known (see [51]).

Proposition 5.8. *Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$ and $q \in (0, \infty]$.*

(i) *Let $p \in (0, \infty]$ and N_1 be sufficiently large (in dependence on s, p, τ). Then $f \in B_{p,q}^{s,\tau}(\mathbb{R}^n)$ if and only if f can be represented as in (5.4) (convergence in $\mathcal{S}'(\mathbb{R}^n)$) and*

$$\|\Phi(f)\|_{b_{p,q}^{s,\tau}(\mathbb{R}^n)}^* := \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{Q_{0,m} \subset P} |\langle f, \phi_{0,m} \rangle|^p \right\}^{1/p} + \sum_{i=1}^{2^n-1} \|\{\langle f, \psi_{i,j,k} \rangle\}_{j,k}\|_{b_{p,q}^{s,\tau}(\mathbb{R}^n)}$$

is finite. In addition, the quasi-norms $\|\Phi(f)\|_{b_{p,q}^{s,\tau}(\mathbb{R}^n)}^$ and $\|f\|_{B_{p,q}^{s,\tau}(\mathbb{R}^n)}$ are equivalent.*

(ii) *Let $p \in (0, \infty)$ and N_1 be sufficiently large (in dependence on s, p, q and τ). Then $f \in F_{p,q}^{s,\tau}(\mathbb{R}^n)$ if and only if f can be represented as in (5.4) (convergence in $\mathcal{S}'(\mathbb{R}^n)$) and*

$$\|\Phi(f)\|_{f_{p,q}^{s,\tau}(\mathbb{R}^n)}^* := \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{Q_{0,m} \subset P} |\langle f, \phi_{0,m} \rangle|^p \right\}^{1/p} + \sum_{i=1}^{2^n-1} \|\{\langle f, \psi_{i,j,k} \rangle\}_{j,k}\|_{f_{p,q}^{s,\tau}(\mathbb{R}^n)}$$

is finite. In addition, the quasi-norms $\|\Phi(f)\|_{f_{p,q}^{s,\tau}(\mathbb{R}^n)}^$ and $\|f\|_{F_{p,q}^{s,\tau}(\mathbb{R}^n)}$ are equivalent.*

Remark 5.9. (i) Such isomorphisms in the framework of Morrey spaces and smoothness spaces related to Morrey spaces are also investigated in [115] ($s > 0$); for $\tau < 1/p$, one may also consult Sawano [75] and Rosenthal [71].

(ii) There is a particular case of Proposition 5.8 which plays a role in our investigations. Let $\tau = 0$ and $p = q = \infty$ (see Proposition 5.2(i)). Then, for $N_1 > |s|$, we have $f \in B_{\infty,\infty}^s(\mathbb{R}^n)$ if and only if f can be represented as in (4.22) (convergence in $\mathcal{S}'(\mathbb{R}^n)$) and

$$\begin{aligned} \|\Phi(f)\|_{b_{\infty,\infty}^s(\mathbb{R}^n)}^* &:= \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{Q_{0,m} \subset P} |\langle f, \phi_{0,m} \rangle| \right\} \\ &+ \sup_{i=1, \dots, 2^n-1} \sup_{j \in \mathbb{Z}_+} 2^{j(s+n/2)} \sup_{k \in \mathbb{Z}^n} |\langle f, \psi_{i,j,k} \rangle| < \infty. \end{aligned}$$

In addition, the quasi-norms $\|\Phi(f)\|_{b_{\infty,\infty}^s(\mathbb{R}^n)}^*$ and $\|f\|_{B_{\infty,\infty}^s(\mathbb{R}^n)}$ are equivalent.

Also Besov-Morrey and Triebel-Lizorkin-Morrey spaces allow such characterizations. However, for Triebel-Lizorkin-Morrey spaces, this follows immediately from Proposition 5.5. Hence, we may concentrate on Besov-Morrey spaces.

Definition 5.10. Let $s \in \mathbb{R}$, $q \in (0, \infty]$ and $0 < p \leq u \leq \infty$. The *sequence space* $n_{u,p,q}^s(\mathbb{R}^n)$ is defined as the space of all sequences $t := \{t_Q\}_{Q \in \mathcal{Q}^*} \subset \mathbb{C}$ such that

$$\|t\|_{n_{u,p,q}^s(\mathbb{R}^n)} := \left\{ \sum_{j \in \mathbb{Z}_+} 2^{j(s+\frac{n}{2})q} \left\| \sum_{\ell(Q)=2^{-j}} |t_Q| \chi_Q \right\|_{\mathcal{M}_p^u(\mathbb{R}^n)}^q \right\}^{1/q} < \infty.$$

Proposition 5.11. *Let $s \in \mathbb{R}$, $p, q \in (0, \infty]$ and $p \leq u \leq \infty$. Let N_1 be sufficiently large (in dependence on s, p, τ). Then $f \in \mathcal{N}_{u,p,q}^s(\mathbb{R}^n)$ if and only if f can be represented as in (4.22) (convergence in $\mathcal{S}'(\mathbb{R}^n)$) and*

$$\|\Phi(f)\|_{n_{u,p,q}^s(\mathbb{R}^n)}^* := \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{Q_{0,m} \subset P} |\langle f, \phi_{0,m} \rangle| \right\} + \sum_{i=1}^{2^n-1} \|\{ \langle f, \psi_{i,j,k} \rangle \}_{j,k}\|_{n_{u,p,q}^s(\mathbb{R}^n)}$$

is finite. In addition, the quasi-norms $\|\Phi(f)\|_{n_{u,p,q}^s(\mathbb{R}^n)}^$ and $\|f\|_{\mathcal{N}_{u,p,q}^s(\mathbb{R}^n)}$ are equivalent.*

For a proof, we refer the reader to [75, 78] and [71].

5.5 Spaces on domains

Spaces on domains are defined by restrictions in our article. For us, this is the most convenient way within this article. Here, for all domains $\Omega \subset \mathbb{R}^n$ and $g \in \mathcal{S}'(\mathbb{R}^n)$, $g|_\Omega$ denotes the restriction of g on Ω .

Definition 5.12. Let $X(\mathbb{R}^n)$ be a quasi-normed space of tempered distributions such that $X(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. Let Ω denote an open, nontrivial subset of \mathbb{R}^n . Then $X(\Omega)$ is defined as the collection of all $f \in \mathcal{D}'(\Omega)$ such that there exists a distribution $g \in X(\mathbb{R}^n)$ satisfying

$$f(\varphi) = g(\varphi) \quad \text{for all } \varphi \in \mathcal{D}(\Omega).$$

Here $\varphi \in \mathcal{D}(\Omega)$ is extended by zero on $\mathbb{R}^n \setminus \Omega$. Moreover, let

$$\|f\|_{X(\Omega)} := \inf \left\{ \|g\|_{X(\mathbb{R}^n)} : g|_\Omega = f \right\}.$$

Hence, the spaces $F_{p,q}^{s,\tau}(\Omega)$, $B_{p,q}^{s,\tau}(\Omega)$, $\mathcal{E}_{u,p,q}^s(\Omega)$ and $\mathcal{N}_{u,p,q}^s(\Omega)$ are now well defined. In this article, we also consider Morrey spaces on domains and Campanato spaces on domains. These spaces are not always spaces of distributions. Therefore we gave in Section 1 of this article and in Subsection 2.3 independent definitions of Campanato spaces and Morrey spaces. They coincide in the sense of equivalent norms in case that both variants are applicable.

Acknowledgements Wen Yuan is supported by the National Natural Science Foundation of China (Grant No. 11471042) and the Alexander von Humboldt Foundation. Dachun Yang is supported by the National Natural Science Foundation of China (Grant Nos. 11171027 and 11361020). This project is also partially supported by the Specialized Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20120003110003) and the Fundamental Research Funds for Central Universities of China (Grant Nos. 2013YB60 and 2014KJCA10). The authors would like to thank Professor Yoshihiro Sawano and Doctor Alberto Arenas Gómez for some helpful discussions on Remarks 2.36 and 2.24, respectively.

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